The H^∞ – Calculus and Sum of Closed Operators

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Abstract

We apply the general operator – valued functional calculus to the joint functional calculus of two sectorial operators . We give the optimal order of smoothness in the Mihlin and Hörmander conditions for operator – valued Fourier multiplier theorems . We give an extended estimate that shows the integral converges to a Bachner integral. More generally we also extended a series of H^∞-sectorial operators has an L_p maximal regularity.

Keywords: $H^\infty -$ calculus, sectorial operators, Banach space, closed operators, srong operator topology

Introduction

The notion of an $H^\infty -$ calculus for sectorial operators on a Banach space has played an important role in spectral theory for unbounded operators and its applications to differential operators and evolution equations. We recall that a sectorial operator of type $0 \le \omega \le \pi$ satisfies a parabolic estimate of type

$$\left\|\zeta R(\zeta, A)\right\| \le C_\sigma \quad \left|\arg\zeta\right| \ge \sigma \qquad (1)$$

for every $\omega < \sigma < \pi$. This estimate allows a definition of $f(A)$ as a bounded operator for functions f which are bounded and analytic on the sector $\sum_\sigma = \{\zeta : |\arg\zeta| < \sigma\}$ and which obey a condition of the type $|f(\zeta)| \le C\left(|\zeta| \Big/ \left(1+|\zeta|^2\right)^\in\right)$ for some $\in > 0$.

If we have an estimate $\|f(A)\| \le C\|f\|_{H^\infty(\sum_\sigma)}$

it is possible to extend the definition of $f(A)$ to any $f \in H^\infty\left(\sum_\sigma\right)$ and we say that f has an $H^\infty\left(\sum_\sigma\right)$- calculus. It is well known that many systems of parabolic differential operators do have an $H^\infty -$ calculus.

Of particular importance are two closely related problems:

The maximal $L_p -$ regularity of the Cauchy problem

$$y'(t) + Ay(t) = f(t), \quad y(0) = 0$$

for a sectorial operator of type $\omega < \pi/2$

-The question whether the sum $A+B$ with domain $D(A) \cap D(B)$ of two sectorial operators is closed.

In fact the first problem can be reduced to the second, and the latter problem is essentially the question whether one can construct a bounded operator $B(A+B)$. This then is a special case of the problem of constructing a joint functional calculus of A, B . In the case of Hilbert spaces and some related situations it was shown that one can construct an operator-valued functional calculus associated to an operator with $H^\infty -$ calculus and this permits a solution; however, it was also shown that such an approach cannot work in general Banach spaces and additional conditions are therefore needed.

We described a method of setting up the joint functional calculus of n sectorial operators and an operator-valued extension. We recall the notion of Rademacher-boundedness ($R-$boundedness) of families of operators. This implicitly goes in connection with vector-valued multiplier theorems. We also introduce some weaker notions and study their relationship to certain Banach space properties of the underlying space.

We prove a very general result on the existence of an operator-valued functional calculus for operators with an $H^{\infty}-$calculus. This permits us to replace boundedness of the range of the function by Rademacher-bounded (or even the weaker concept of U-boundedness).

We study the relationship between R-boundedness and the functional calculus for general sectorial operators of particular importance is the notion of R-sectoriality when the boundedness condition (1) is replaced by an R-boundedness condition.

We show that if A, B are sectorial operators such that A has an $H^{\infty}\left(\sum_{\sigma}\right)$ -calculus and B is R- sectorial of type σ' when $\sigma+\sigma'<\pi$ then $A+B$ (with domain $D(A)\cap D(B)$) is closed.

One advantage of this result is that it is easier to check R-sectoriality than the boundedness of imaginary powers.

We give applications to the joint $H^{\infty}-$functional calculus and show how Banach space properties such as UMD, analytic UMD and property (α) of Pisier relate to the results.

$H^{\infty}-$calculus really induces an unconditional expansion of the identity of the underlying Banach space.

We use this observation to show how classical results on unconditional bases can be recast as results on operators with $H^{\infty}-$calculus on L_1 and $C(k)-$spaces, but they are in practice very few examples of such operators of this type.

Here we sketch a method of setting up an operator-valued functional calculus for finite collections of sectorial operators.

Let us first introduce some notion. Suppose $0<\sigma<\pi$. Then we denote by $\sum_{\sigma}$ the sector $\{z:|\arg|<\sigma,|z|>0\}$ and by Γ_{σ} the contour $\left\{|t|e^{i(\operatorname{sgn} t)\sigma}:-\infty<t<\infty\right\}$. We denote by $H^{\infty}\left(\sum_{\sigma}\right)$ the space of all bounded analytic functions on $\sum_{\sigma}$. We define $H_0^{\infty}\left(\sum_{\sigma}\right)$ to be the subspace of all $f\in H^{\infty}\left(\sum_{\sigma}\right)$ which obey an estimate of the form $|f(z)|\le c\left(|z|/\left(1+|z|^2\right)\right)^{\epsilon}$ with $\epsilon>0$. Let us extend this to dimension m. In $\mathbb{C}^m$ if $\sigma=(\sigma_1,...,\sigma_m)$ where $0<\sigma_k<\pi$ we define $\sum_{\sigma}=\Pi_{k=1}^{m}\sum_{\sigma_k}$ and $\Gamma_{\sigma}=\Pi_{k=1}^{m}\Gamma_{\sigma_k}$. If $\sigma, v\in\mathbb{C}^m$ we write $\sigma>v$ if $\sigma_k>v_k$ for $1\le k\le m$. We denote by $H^{\infty}\left(\sum_{\sigma}\right)$ to be the space of all bounded analytic functions on $\sum_{\sigma}$. We define $H^{\infty}\left(\sum_{\sigma}\right)$ to be the subspace of all $f\in H^{\infty}\left(\sum_{\sigma}\right)$ which obey an estimate of the form $|f(z)|\le C\Pi_{k=1}^{m}\left(|z_k|/\left(1+|z_k|^2\right)\right)^{\varepsilon}$ with $\varepsilon>0$ where $z=(z_1,...,z_m)$.

We introduce some corresponding vector valued spaces. Now suppose X is a Banach space and A is sub-algebra of $L(X)$, which is closed for the strong-operator topology. If $\sigma=(\sigma_1,...,\sigma_m)$ as above, we define $H^{\infty}\left(\sum_{\sigma};A\right)$ the space of all bounded functions $F:\sum_{\sigma}\to A$, so that for every $x\in X$ the map $z\to F(z)x$ is analytic (i.e., F is analytic for some strong-operator topology. We consider the scalar space $H^{\infty}\left(\sum_{\sigma}\right)$ as a subspace of $H^{\infty}\left(\sum_{\sigma},A\right)$ via the identification $f\to fI$.

We shall say that F_n converges boundedly to F in $H^\infty\left(\sum_\sigma, A\right)$ if $\sup_n \sup_{z\in\sum_\sigma} \left\|F_n(z)\right\| < \infty$ and $F_n(z)x \to A(z)x$ for every $z \in \sum_\sigma$, and $x \in X$. We define $H_0^\infty\left(\sum_\sigma; A\right)$ the subspace of all $F \in H^\infty\left(\sum_\sigma; A\right)$ which obey an estimate of the form $\left\|F(z)\right\| \le C\prod_{k=1}^m \left(|z_k| \big/ \left(1+|z_k|^2\right)\right)^\in$ with $\varepsilon > 0$ where $z = (z_1,\dots,z_m)$.

We next consider the space of germs of such functions. Fix $0 \le \omega_k < \pi$ for $1 \le k \le m$. We consider the space $H(\omega, A) = \bigcup_{\sigma>w} H^\infty\left(\sum_\sigma; A\right)$ where (F, G) are identified if there exists $\sigma > \omega$ will $F(z) = G(z)$ for all $z \in \sum_\sigma$. $H(\omega, A)$ is then an algebra. In $H(\omega, A)$ we define a notion of sequential convergence τ by $F_n \to F$ if there exists $\sigma > \omega$ so that each $F_n, F \in H^\infty\left(\sum_\sigma; A\right)$, $\sup_n \sup_{z\in\sum_\sigma} \left\|F(z)\right\| < \infty$ and $F_n(z)x \to F(z)x$ for all $z \in \sum_\sigma$ and all $x \in X$.

Recall that a closed densely defined operator A on a Banach space X is a sectorial operator of type $0 \le \omega = \omega(A) < \pi$ if A is one-one with dense range, the resolvent $R(\lambda, A)$ is defined and bounded for $\lambda = re^{i\theta}$ where $r > 0$ and $\omega < |\theta| \le \pi$ and satisfies an estimate $\left\|\lambda R(\lambda, A)\right\| \le C_\sigma$ for $\omega < \sigma \le |\theta|$.

Suppose $(A_1,\dots,A_m)$ is a family of sectorial operators where A_k is of type ω_k for $1 \le k \le m$, and let $\omega = (\omega_1,\dots,\omega_m)$. Define the resolvent for $|\arg\lambda| > \omega$ by $R(\lambda, A_1,\dots,A_m) = \prod_{k=1}^m R(\lambda_k, A_k)$. Let A be the closed sub-algebra of $\mathrm{L}(X)$ of all operators T so that T commutes with $R(\lambda, A_k)$ for every k and every λ with $|\arg\lambda| > \omega_k$.

If $F \in H(\omega, A)$ is of the form $F(z) = \prod_{k=1}^m (\lambda_k - z_k)^{-p_k S}$ where $p_k \in \square \bigcup\{0\}$ and $S \in A$ we define $F(A_1,\dots,A_m) = \prod_{k=1}^m R(\lambda_k, A_k)^{p_k S}$ and then this definition can be extended by linearity to the linear span of such functions, which we call the rational functions, denoted $R(\omega, A)$, in $H(\omega, A)$.

To extend this definition further we use the following device. Consider the algebra of all $\left(F, F(A_1,\dots,A_m)\right)$ for $F \in R(\omega, A)$ as a subset of $H(\omega, A)\times A$. Denote by τ^* the sequential convergence $(F_n, T_n) \to (F, T)$ if $F_n \to F(\tau)$ and $T_n \to T$ in the strong-operator topology. Let B be the $\tau^* -$ closure of this set (i.e. the smallest set which is closed under sequential convergence and contains it). Notice that this construction might involve taking infinitely many iterations of sequential limits, but our construction actually shows that two iterations suffice. It is clear that B is an algebra. Our next task is to show that if $F \in H(\omega, A)$ there is at most one choice of $T \in A$ so that $(F, T) \in B$, this will enable us to define $F(A_1,\dots,A_m)$ unambiguously.

Consider the function on C

$$\varphi_n(z) = \frac{n}{n+z} - \frac{1}{1+nz} \tag{2}$$

and then define on C^m, $\psi_n(z) = \prod_{k=1}^m \varphi_n(z_k)$ so that $\psi_n \in H_0^\infty\left(\sum_\sigma\right)$ for every $\sigma > 0$. Then

$\psi_n(A_1,\dots,A_m) = \prod_{k=1}^m \left(\frac{1}{n} R\left(-\frac{1}{n}, A_k\right) - nR(-n, A_k)\right) = V_n$ is an approximate identity in the sense that $\sup\left\|V_n\right\| < \infty$ and $V_n x \to x$ for every $x \in X$ if $F \in H(\omega, A)$ then if $F \in H^\infty\left(\sum_\sigma, A\right)$ we can define

$$L_n(F)x = \left(\frac{-1}{2\pi i}\right)\int_{\Gamma_v} \psi_n(\zeta)F(\zeta)R(\zeta, A_1,...,A_m)xd\zeta\ , \tag{3}$$

as long as $\omega < v < \sigma$. (This is multiple contour integral). An application of Cauchy's Theorem shows that L_n is independent of the choice of v . By the Lebesgue Dominated convergence Theorem $L_n : H(\omega, A) \to A$ is $\tau -$ continuous if A is equipped with the strong-operator topology.

If F is rational then we have by a standard contour integration,

$$L_n(F)x = F(A_1,...,A_m)V_n x \qquad x \in X \tag{4}$$

Now the map $(F,T) \to L_n(F) - T$ is continuous for τ^* and the strong-operator topology. We conclude that if $(F,T) \in B$,

$L_n(F)x = TV_n x \quad x \in X$.

Since $V_n x \to x$ for all $x \in X$, this shows that T is uniquely determined by F . Hence we can define $H(A_1,..., A_m; A)$ to be the set of $F \in H(\omega, A)$ such that for some T we have $(F,T) \in B$ and then we can define $T = F(A_1,..., A_m)$ for $F \in H(A_1,..., A_m; A)$. The space $H(A_1,..., A_m; A)$ is an algebra and $F \to F(A_1,..., A_m)$ is an algebra homomorphism.

For $F \in H(A_1,..., A_m; A) \cap H^\infty\left(\sum_\sigma \ ; A\right)$ and $\sigma > v > \omega$ then (3) and (4) can be written as :

$$F(A_1,..., A_m)V_n x = \left(\frac{-1}{2\pi i}\right)^m \int_{\Gamma_v} \psi_n(\zeta)R(\zeta, A_1,..., A_m)xd\zeta \tag{5}$$

If $F \in H_0^\infty\left(\sum_\sigma \ , A\right)$ then the integral in (5) converges as $n \to \infty$. We can show by approximating the integral by Riemann sums that $F \in H(A_1,..., A_m; A)$ and then we have:

$$F \in H(A_1,..., A_m)x = \left(\frac{-1}{2\pi i}\right)^m \int_{\Gamma_v} F(\zeta)R(\zeta, A_1,..., A_m)xd\zeta \tag{6}$$

It now follows that if $F \in H(\omega, A)$ then $(\psi_k F) \in H(A_1,..., A_m; A)$ for each $k \in N$. Furthermore if $F_n \to F(\tau)$ we have $(\psi_k F_n)(A_1,..., A_m) \to (\psi_k F)(A_1,..., A_m)$ in the strong-operator topology for each fixed k . From this it follows that if $F_n \in H(A_1,..., A_m; A)$ and $\sup \|F_n(A_1,..., A_m)\| < \infty$ then $F \in H(A_1,..., A_m)$ and $F_n(A_1,..., A_m) \to F(A_1,..., A_m)$ in the strong-operator topology (we have convergence on each $V_n x$). In particular it follows that $F \in H(A_1,..., A_m; A)$ if and only if $\sup_n \|(\psi_n F)(A_1,..., A_m)\| < \infty$.

If we consider the scalar functions in $H(A_1,..., A_m) \subset H(A_1,..., A_m; A)$ then we have defined the joint functional calculus for $(A_1,.., A_m)$. We recall that a single operator A has an $H^\infty\left(\sum_\sigma \ \right)$ - Calculus if $H^\infty\left(\sum_\sigma \ \right) \subset H(A)$. The collection $(A_1,.., A_m)$ has a joint $H^\infty\left(\sum_\sigma \ \right)$-calculus if $H^\infty\left(\sum_\sigma \ \right) \subset H(A_1,..., A_m)$.

We recall that a family F of bounded operators on a Banach space X is called Rademacher-bounded or R-bounded with R-boundedness constant C if letting $(\varepsilon_k)_{k=1}^\infty$ be a sequence of independent Rademachers on some probability space then for every $x_1,..., x_n \in X$ and $T_1,..., T_n \in F$ we have:

$$\left(E\left\| \sum_{k=1}^{n} \varepsilon_k T_k x_k \right\|^2 \right)^{\frac{1}{2}} \le C\left(E\left\| \varepsilon_k x_k \right\|^2 \right)^{\frac{1}{2}} \qquad (7)$$

It is important to note that this definition and the associated constant C are unchanged if we require $T_1,...,T_n$ to be distinct in (7).

We will introduce two related weak notions. Let that F is weakly Rademacher-bounded or WR-bounded with WR-boundedness constant C if for every $x_1,...,x_n \in X$, $x_1^*,...,x_n^* \in X^*$ and $T_1,...,T_n \in F$ we have

$$\sum_{k=1}^{n} \left| \left\langle T_k x_k , x_k^* \right\rangle \right| \le CE\left(\left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|^2 \right)^{\frac{1}{2}} E\left(\left\| \sum_{k=1}^{n} \varepsilon_k x_k^* \right\|^2 \right)^{\frac{1}{2}} \qquad (8)$$

Finally we say that F is unconditionally bounded or U-bounded with U-boundedness constant C if for every $x_1,...,x_n \in X$, $x_1^*,...,x_n^* \in X^*$ and $T_1,...,T_n \in F$ we have

$$\sum_{k=1}^{n} \left| \left\langle T_k x_k , x_k^* \right\rangle \right| \le \max_{\varepsilon_k=\pm1} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\| \max_{\varepsilon_k=\pm1} \left\| \sum_{k=1}^{n} \varepsilon_k x_k^* \right\| \qquad (9)$$

Lemma (1.1)[3][1]:

Let F be a subset of $L(X)$. Then for F , R-bounded $\Rightarrow$ WR-bounded $\Rightarrow$ U-bounded. If X has nontrivial Rademacher type then WR-bounded $\Rightarrow$ R-bounded.

The last sentence is the non-trivial part of the lemma and this follows easily from Pisier's characterization of spaces with non-trivial type as those in which the Rademacher projection is bounded.

We shall also need some related Banach space concepts. Suppose $(\varepsilon_k)_{k=1}^{\infty}$ and $(\eta_k)_{k=1}^{\infty}$ are two mutually independent sequences of Rademachers. We say that X has property (α) if there is a constant C so that for any $(x_{jk})_{j,k=1}^{n} \subset X$ and for any $(\alpha_{ik})^{n}_{j,k=1} \subset C$ we have

$$\left(E\left\| \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} \varepsilon_j \eta_k x_{jk} \right\|^2 \right)^{1/2} \le \max{}_{j,k}^{C_x} \left| \alpha_{jk} \right| \left(E\left\| \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_j \eta_k x_{jk} \right\|^2 \right)^{1/2} \qquad (10)$$

We say that X has property (A) if there is a constant C such that for any $(x_{j,k})_{j,k=1}^{n} \subset X$ and for any $(x^*{}_{jk})_{j,k=1}^{n} \subset X^*$ we have

$$\sum\sum \left| \left\langle x_{jk} , x_{jk}^* \right\rangle \right| \le C\left(E\left\| \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_j \eta_k x_{jk} \right\|^2 \right)^{\frac{1}{2}} \left(E\left\| \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_j \eta_k x_{jk}^* \right\|^2 \right)^{\frac{1}{2}} \qquad (11)$$

Clearly $(\alpha) \Rightarrow$ (A) and the converse holds if X has nontrivial Rademachers type; this is a fairly simple deduction from the boundedness of the Rademacher projection. Any subspace of a Banach Lattice with nontrivial co type has property (α) while any Banach lattice has property (A). It is also observed that L_1/H_1 has (α). The Schatten ideals C_p when $1 \le p \le \infty$ and $p \ne 2$ fail to have (A).

We shall say that X has property (Δ) if there is a constant C so that for any $(x_{j,k})_{j,k=1}^{n} \in X$

$$\left(E\left\|\sum_{j=1}^{n}\sum_{k=1}^{j}\varepsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}} \le C\left(E\left\|\sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}} \qquad (12)$$

It is clear that (Δ) is a weaker property than (α). It is in fact shared by all spaces with (UMD) and even analytic UMD. We recall that X has analytic UMD if every L_1–bounded analytic martingale has unconditional martingale differences.

Proposition (1.2)[3][4]:

Suppose X has analytic UMD. Then X has property (Δ).

Proof:

Let $(\tilde{\varepsilon}_k)_{k=1}^{\infty}$ and $(\tilde{\eta}_k)_{k=1}^{\infty}$ be two mutually independent sequences of Steinhaus variables (i.e., each is complex-valued and uniformly distributed on the unit circle). By applying the unconditionality of the Rademachers and the Khintchine-Kahane inequality it is sufficient to show the existence of a constant C so that for any $(x_{jk})_{j,k=1}^{\infty}$ we have

$$E\left\|\sum_{j=1}^{n}\sum_{k=1}^{j}\tilde{\varepsilon}_j\tilde{\eta}_k x_{jk}\right\| \le CE\left\|\sum_{j=1}^{n}\sum_{k=1}^{n}\tilde{\varepsilon}_j\tilde{\eta}_k x_{jk}\right\|$$

To see this we define f_j for $1 \le j \le 2n-1$ by $f_{2r-1} = \sum_{j\le r}\sum_{k\le r}\tilde{\varepsilon}_j\tilde{\eta}_k x_{jk}$ and $f_{2r} = \sum_{j\le r+1}\sum_{k\le r}\tilde{\varepsilon}_j\tilde{\eta}_k x_{jk}$ .Let $f_0 = 0$. Then (f_j) is an analytic martingale and so for a suitable constant C depending only on X we have:

$$\mathrm{E}\left\|\sum_{r=0}^{n-1}(f_{2r-1} - f_{2r})\right\| \le CE\left\|f_{2n-1}\right\|$$

This yields the desired inequality. Since any space with (UMD) has analytic (UMD) this shows that (UMD) spaces have (Δ); actually a direct proof using Rademacher in place of Steinhans variables in the above argument is possible for this case. Thus the Schatten classes C_p have property (Δ) as long as $1 < p < \infty$. However Haagerup and Pisier show that C_1 (which has co type 2) fails analytic UMD and their argument actually shows it fails property (Δ). This implies that $C(k)$-spaces of infinite dimension also fail (Δ) since C_1 is finitely representable in any such space.

Theorem (1.3)[3]:

Suppose $(U_k)_{k=1}^{\infty}$ and $(V_k)_{k=1}^{\infty}$ are two sequences of operators in $L(X)$ satisfying

$$\sup_n \sup_{\varepsilon_k=\pm 1}\left\|\sum \varepsilon_k U_k\right\| \le M < \infty$$

And

$$\sup_n \sup_{\varepsilon_k=\pm 1}\left\|\sum \varepsilon_k V_k\right\| \le M < \infty$$

Suppose further $\mathcal{F} \subset L(X)$ is a family of operators which is R-bounded with constant R. Then

(i) The sequence $(U_k)_{k=1}^{\infty}$ is R-bounded with constant M.

(ii) If X has property (α) the collection $\left\{\sum_{k=1}^{n}\alpha_k U_k T_k V_k : n \in N, |\alpha| \le 1, T_k \in \mathcal{F}\right\}$ is R-bounded with constant CRM^2 where C depends only on X.

(iii) If X has property (A) then the family $\left\{\sum_{k=1}^{n}\alpha_k U_k V_k : n\in N, |\alpha_1|,...,|\alpha_n|\le 1\right\}$ is WR-bounded with constant CM^2 where C depends only on X.

(iv) If X has property (Δ) then the set $\left\{\sum_{k=1}^{n}U_k V_k : n\in N\right\}$ is R-bounded with constant CM^2 where C depends only on X.

Proof:

(i) We use the remark that it is enough to establish (7) for distinct operators $T_1,...,T_n$. If $x_1,...,x_n\in X$ and $\alpha_k=\pm 1$ then

$$E\left(\left(\sum_{k=1}^{n}\varepsilon_k U_k\right)\left(\sum_{k=1}^{n}\varepsilon_k\alpha_k x_k\right)\right)=\sum_{k=1}^{n}\varepsilon_k U_k x_k \text{ and hence}$$

$$\sup_{\alpha_k=\pm1}\left\|\sum_{k=1}^{\infty}\varepsilon_k U_k x_k\right\|\le M\left(E\left(\left\|\sum_{k=1}^{n}\varepsilon_k X_k\right\|^2\right)\right)^{1/2}.$$

This proves (i) and indeed a rather stronger result.

Let $S_j=\sum_{k=1}^{\infty}\alpha_{jk}U_k T_{jk}V_k$ where $T_{jk}\in F$ and $\left(\alpha_{jk}\right)$ is a finitely nonzero collection of complex numbers with (ii)

$\left|\alpha_{jk}\right|\le 1$ and fix $x_1,...,x_n\in X$. We first note that for all $y_1,...,y_n\in Y$.

$$\left\|\sum_{k=1}^{n}U_k y_k\right\|=\left\|E\left(\left(\sum_{k=1}^{n}\varepsilon_k U_k\right)\left(\sum_{k=1}^{n}\varepsilon_k y_k\right)\right)\right\|\le M\left(E\left\|\sum_{k=1}^{n}\varepsilon_k y_k\right\|^2\right)^{1/2}$$

We will also use the fact that there is a constant C depending only on X so that for $\left(\chi_{jk}\right)_{j,k=1}^{n}\in X$ we have from property (α),

$$\left(E_\varepsilon E_\eta\left\|\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_{jk}\varepsilon_j\eta_k T_{jk}x_{jk}\right\|^2\right)^{\frac{1}{2}}\le CR\left(E_\varepsilon E_\eta\left\|\sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}}$$

Hence, using $y_k=\sum_{j=1}^{n}\alpha_{jk}\varepsilon_j T_{jk}V_k x_j$ in the first inequality,

$$\begin{aligned}\left(E\left\|\sum_{j=1}^{n}\varepsilon_j S_j x_j\right\|^2\right)^{\frac{1}{2}}&=\left(E\left\|\sum_{k=1}^{\infty}U_k\sum_{j=1}^{n}\alpha_{jk}\varepsilon_j T_{jk}V_k x_j\right\|^2\right)^{\frac{1}{2}}\\&\le M\left(E_\varepsilon E_\eta\left\|\sum_{j=1}^{n}\sum_{k=1}^{\infty}\alpha_{jk}\varepsilon_j\eta_k T_{jk}V_k x_j\right\|^2\right)^{\frac{1}{2}}\\&\le CRM\left(E_\varepsilon E_\eta\left\|\sum_{j=1}^{n}\sum_{k=1}^{\infty}\varepsilon_j\eta_k V_k x_j\right\|^2\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq CRM^2\left(E_\varepsilon\left\|\sum_{k=1}^{\infty}\varepsilon_j x_j\right\|^2\right)^{\frac{1}{2}}$$

This proves (ii).

(i) Let $S_j=\sum_{k=1}^{\infty}\alpha_{jk}U_kV_k$ where (α_{jk}) is a finitely matrix with $|\alpha_{jk}|\leq 1$. In this case if $x_1,\dots,x_n\in X$ and $x_1^*,\dots,x_n^*\in X^*$ we note that:

$$\sum_{j=1}^{n}\left|\left\langle S_jx_j,x_j^*\right\rangle\right|\leq\sum_{j=1}^{n}\sum_{k=1}^{\infty}\left|\left\langle V_xx_j,U_k^*x_j^*\right\rangle\right|$$

$$\leq C\left(E_\varepsilon E_\eta\left\|\sum_{j=1}^{n}\sum_{k=1}^{\infty}\varepsilon_j\eta_kV_kx_j\right\|^2\right)^{\frac{1}{2}}\left(E_\varepsilon E_\eta\left\|\sum_{j=1}^{n}\sum_{k=1}^{\infty}\varepsilon_j\eta_kU_k^*x_j^*\right\|^2\right)^{\frac{1}{2}}$$

$$\leq CM^2\left(E\left\|\sum_{j=1}^{n}\varepsilon_jx_j\right\|^2\right)^{\frac{1}{2}}\left(E\left\|\sum_{j=1}^{n}\varepsilon_jx_j^*\right\|^2\right)^{\frac{1}{2}}$$

(i) We use the proof of (ii). This time we again use the fact it suffices to consider the operators without repetition. So we consider $S_j=\sum_{k=1}^{j}U_kV_k$ and repeat the proof of (ii) with $\alpha_{jk}=1$ if $k\leq j$ and 0 otherwise and replace each T_{jk} by the identify. Using (12) in place of (10) gives the desired conclusion.

Lemma (1.4)[3]:

Suppose $0<\sigma<\pi$ and $F\in H^\infty\left(\sum_\sigma;\mathcal{L}\left(X\right)\right)$. Suppose $0<\sigma_0<v<\sigma$ and for some $M<\infty$ and $a>1$, and for each $t\in R$ the set $\left\{F\left(a^kte^{\pm iv}\right)\right\}_{k\in z}$ is $U-$bounded (respectively, WR-bounded; respectively, R-bounded) with constant bounded by M (independent of t). Then the family $\left\{F\left(Z\right):z\in\sum_{\sigma_0}\right\}$ is U-bounded, (respectively, WR-bounded; respectively R-bounded).

Proof: We give the proof in the U-boundedness case, the others being similar. We first make the observation that it suffices to consider the case when $v=\pi/2$ as one can make the transformation $z=\omega^{2v/\pi}$. In this case we have the formula

$$F\left(z\right)=\frac{1}{\pi}\int_{-\infty}^{\infty}F\left(it\right)\Re\left(z-it\right)^{-1}dt$$

We write

$$F_1\left(z\right)=\frac{1}{\pi}\int_{0}^{\infty}F\left(it\right)\Re\left(z-it\right)^{-1}dt$$

and

$$F_2\left(z\right)=\frac{1}{\pi}\int_{0}^{\infty}F\left(-it\right)\Re\left(z+it\right)^{-1}dt$$

so that $F\left(z\right)=F_1\left(z\right)+F_2\left(z\right)$

Note that for a suitable constant C we have an estimate $0 \le R\left(z \pm it\right)^{-1} \le C\left|z\right|\min\left(t^{-2},\left|z\right|^{-2}\right)$ whenever $z \in \sum_{\sigma_0}$.

Now suppose $x_1,...,x_n \in X$ and $x_1^*,...,x_n^* \in X^*$. Suppose $z_1,...,z_n \in \sum_{\sigma_0}$. Let us suppose $m_j \in \mathbb{Z}$ are chosen so that $a^{m_j} \le \left|z_j\right| \le a^{m_j+1}$. We have for $z \in \sum_{\sigma_0}$

$$\sum_{j=1}^{n}\left|\left\langle F_1\left(z_j\right)x_j, x_j^*\right\rangle\right| \le \frac{1}{\pi}\sum_{j=1}^{n}\int_0^\infty\left|\left\langle F\left(ia^{m_j}t\right)x_j, x_j^*\right\rangle\right| R\left(z_j - ia^{m_j}t\right)x$$

$$\le \frac{ac}{\pi}\sum_{j=1}^{n}\int_0^\infty\left|\left\langle F\left(ia^{m_j}t\right)x_j, x_j^*\right\rangle\right|\min\left(1,t^{-2}\right)dt$$

$$\le C\max_{\varepsilon_j=\pm1}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|\max_{\varepsilon_j=\pm1}\left\|\sum_{j=1}^{n}\varepsilon_j x_j^*\right\|$$

for a suitable constant C^1. A similar argument can be done for F_2.

Let us suppose A is sectorial of type ω and $\sigma > \omega$. We let $\mathcal{A}$ denote the algebra of all bounded operators which commute with A .

Before we prove the basic estimate for an operator-valued functional calculus, we will describe in the following lemma and proposition the connection between the H^∞ -calculus and unconditional expansions in the underlying Banach space.

Lemma (1.5)[3]:

Suppose that A admits an $H^\infty \times\left(\sum_\sigma\right)-$ calculus, and that $f \in H_0^\infty\left(\sum_\sigma\right)$. Then there is a constant C so that for any $t > 0$ and any finitely nonzero sequences $\left(\alpha_k\right)_{k\in\mathbb{Z}}$ we have:

$$\left\|\sum_{k\in\mathbb{Z}}\alpha_n f\left(2^k tA\right)\right\| \le C\max_{k\in\mathbb{Z}}\left|\alpha_k\right|$$

Furthermore for every $x \in X$ and $t > 0$ the series $\sum_{k\in\mathbb{Z}} f\left(2^t tA\right)x$ converges unconditionally in X .

Proof: We can assume $\max_{k\in\mathbb{Z}}\left|\alpha_k\right| \le 1$. For suitable constants C, C' and $t > 0$ we have

$$\left\|\sum\alpha_k f\left(2^k tA\right)\right\| \le C\sup_{z\in\Sigma_\sigma}\left|f\left(2^k z\right)\right| \le CC'\sup_{z\in\Sigma_\sigma}\sum_{k\in\mathbb{Z}}\left(\frac{2^k z}{1+2^k\left|z\right|^2}\right)^\varepsilon$$

and the last quantity is finite.

For the last part observe that for any bounded sequence $\left(\alpha_k\right)_{k\in\mathbb{Z}}$ and $t > 0$, the series $\sum_{k\in\mathbb{Z}}\alpha_k f\left(2^k tA\right)x$ must converge to $g\left(A\right)x$ where

$$g\left(z\right) = \sum_{k\in\mathbb{Z}}\alpha_k f\left(2^k tz\right) \in H^\infty\left(\Sigma_\sigma\right).$$

Proposition (1.6)[3]:

Suppose $F \in H_0^\infty\left(\Sigma_\sigma;\mathcal{A}\right)$. Then for any $\omega < \nu < \sigma$, $0 < s < 1$, and any $x \in X$,

$$F\left(A\right)x = \frac{-1}{2\pi i}\int_{\Gamma_\nu}\zeta^{-s}F\left(\zeta\right)A^s R\left(\zeta, A\right)x d\zeta ,$$

Proof: First note that $A^s R(\lambda, A)$ is a bounded operator for $\lambda \in \Gamma_v$ which is given by the integral

$$A^s R(\lambda, A)k = \frac{-1}{2\pi i}\int_{\Gamma_v} \zeta^s (\lambda - \zeta)^{-1} R(\zeta, A) x d\zeta \qquad (13)$$

If $\omega < v' < v$. This gives an estimate $\left\| A^s R(\lambda, A) \right\| \le C_s |\lambda|^{s-1}$ and shows that the integral in (13) converges to a Buchner integral.

It is clear that we only need to establish the formula if $x = \varphi_n(A) y$ for some $y \in X$. To do this we compute

$$\begin{aligned} F(A)\varphi_n(A)x &= F(A)\varphi_n^2(A)y \\ &= A^2\varphi_n(A)\big(F(A)\big)A^{-s}\varphi_n(A)y \\ &= \frac{-1}{2\pi i}\big(A^s\varphi_n(A)\big)\int_{\Gamma_v}\zeta^{-s}\varphi_n(\zeta)F(\zeta)R(\zeta,A)yd\zeta \\ &= \frac{-1}{2\pi i}\int_{\Gamma_v}\zeta^{-s}\varphi_n(\zeta)\mathrm{F}(\zeta)\big(A^s\varphi_n(A)\big)R(\zeta,A)yd\zeta \\ &= \frac{-1}{2\pi i}\int_{\Gamma_v}\zeta^{-s}\varphi_n(\zeta)\mathrm{F}(\zeta)\big(A^sR(\zeta,A)\big)xd\zeta \end{aligned}$$

Now using the dominated convergence theorem we obtain (13).

Let us rewrite (13) by using the parameterization $\zeta = |t| e^{i(sgnt)v}$ for $-\infty < t < \infty$. To represent the resolvent we often use the function.

$$h_s^\rho(z) = z^s\left(e^{i\varphi} - z\right)^{-1} \qquad (14)$$

Then for $F \in H_0^\infty\left(\sum_\sigma; A\right)$ where $\sigma > v > \omega$,

$$\begin{aligned} F(A)x &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i(1-s)(\mathrm{sgn}\,t)v}|t|^{-s} F\left(|t|e^{i(\mathrm{sgn}\,t)v}\right)A^sR\left(|t|e^{i(\mathrm{sgn}\,t)v}, A\right)xdt \\ &= \frac{e^{i(1-s)v}}{2\pi i}\int_0^\infty F\left(te^{iv}\right)h_s^v\left(t^{-1}A\right)x\frac{dt}{t} + \frac{e^{-i(1-s)v}}{2\pi i}\int_0^\infty F\left(te^{-iv}\right)h_s^{-v}\left(t^{-1}A\right)x\frac{dt}{t} \end{aligned}$$

This can be reformulated as:

$$F(A)x = \frac{1}{2\pi i}\int_1^2\left(M_+(t) + M_-(t)\right)\frac{dt}{t} \qquad (15)$$

Where

$$M_\pm(t) = e^{\pm i(1-s)v}\sum_{k\in\mathbb{Z}} F\left(2^{-k}t^{-1}e^{\pm iv}\right)h_s^{\pm v}\left(2^k tA\right)x \qquad (16)$$

Corollary (1.7)[6]:

Suppose $F_j \in (\sum_\sigma, \mathcal{A})$, is a sequence ,with $\omega < \nu < \sigma$, $0 < s < 1$ and $x \in X$.

Then

$$\sum_{j=1}^\infty\sum_{j=1}^\infty F_j(A)\,x = \frac{-1}{2\pi i}\int_\Gamma \sum_{j=1}^\infty \xi^{-s}F_j(\xi)(A_1^s + \cdots + A_1^2)R(\xi, A_1 + \cdots + A_1)xd\xi$$

$$= \frac{-1}{2\pi i}\sum_{j=1}^\infty\sum_{j=1}^\infty \int_{\Gamma_\omega}\xi^{-s}F_j(\xi)A_K^sR(\xi, A_K)xd\xi$$

Proof: We have

$$\sum_{K=1}^{\infty} A_K^s R(\lambda, A_K) L = \frac{-1}{2\pi i} \sum_{K=1}^{\infty} \int_{\Gamma_\omega} \xi^2 (\lambda - \xi)^{-1} R(\xi, A_K) x d\xi$$

For $\omega < \nu' < \nu$. To find the estimate, we have

$$\left| \sum_{K=1}^{\infty} A_K^s R(\lambda, A_K) L \right| = \frac{-1}{2\pi i} \left| \sum_{K=1}^{\infty} \int_{\Gamma_\omega} \xi^s (\lambda - \xi)^{-1} R(\xi, A_K) x d\xi \right|$$

Then

$$\sum_{K=1}^{n} \| A_K^s R(\lambda, A_K) \| \le C_s |\lambda|^{\nu - 1}$$

If $x = \sum_{K=1}^{\infty} \varphi_n(A_K) y$, $y \epsilon X$. We have $\sum_{j=1}^{\infty} \sum_{K=1}^{\infty} F_j(A_K) \varphi_n(A_K) x =$
$\sum_{j=1}^{\infty} \sum_{K=1}^{\infty} F_j(A_K) \varphi_n^2(A_K) y =$
$\sum_{K=1}^{\infty} \sum_{j=1}^{\infty} A_K^2 \varphi_n(A_K) \left(F_j(A_K) \right) A_K^{-s} \varphi_n(A_K) y = \frac{-1}{2\pi i} \sum_{K=1}^{\infty} \left(A_K^s \varphi_n(A_K) \right) \sum_{j=1}^{\infty} \int_{\Gamma_\omega} \xi^{-s} \varphi_n(\xi) R(\xi, A_K)\, y d\xi =$
$\frac{-1}{2\pi i} \sum_{j=1}^{\infty} \sum_{K=1}^{\infty} \int_{\Gamma_\omega} \xi^{-s} \varphi_n(\xi) F_j(\xi) \left(A_K^s \varphi_n(A_K) \right) R(\xi, A_K) y d\xi =$
$\frac{-1}{2\pi i} \sum_{j=1}^{\infty} \sum_{K=1}^{\infty} \int_{\Gamma_\omega} \xi^{-s} \varphi_n(\xi) F_j(\xi) \left(A_K^s R(\xi, A_K) \right) x d\xi$

Proposition (1.6) gives the proof.

We first make an essentially trivial deduction characterizing for the $H^\infty -$ calculus.

Proposition (1.8)[3]:

Suppose $\nu > \omega$ and $0 < s < 1$. Consider the conditions:

$$\sup_{t>0} \sup_{N} \sup_{\varepsilon_k = \pm 1} \left\| \sum_{k=-N}^{N} \varepsilon_k \left(2^k t \right)^{(1-s)} A^s R\left(2^k t e^{\pm i\nu}, A \right) \right\| < \infty \qquad (17)$$

Then (17) is necessary for A to admit an $H^\infty\left(\sum_\sigma\right) -$ calculus for some $\sigma < \gamma$ and sufficient for A to admit an $H^\infty\left(\sum_\sigma\right) -$ calculus for every $\sigma > \gamma$.

Proof:

Necessity follows immediately from Lemma (1.5) for the functions $h_s^{\pm\nu}$. Conversely by (17), if $f \in H^\infty\left(\sum_\sigma\right)$ where $\sigma > \nu$ we obtain by (15) and (16)

$\left\| \left(\varphi_n f \right)(A) \right\| \le C$ independent of n. This implies that $f \in \mathrm{H}(A)$.

The main result is also easy from (15) and (16).

Theorem (1.9)[3][1][4]:

Suppose A admits an $H^\infty\left(\sum_\sigma\right) -$ calculus and $F \in H^\infty\left(\sum_\rho ; A\right)$ for some $\rho > \sigma$. Suppose further that the set $\left\{ F(z) : z \in \sum_\rho \right\}$ is U-bounded. Then $F \in H(A, A)$.

The theorem holds if we assume the stronger property that $\left\{ F(z) : z \in \sum_\rho \right\}$ is WR-bounded or R-bounded.

Proof:

As before we consider $\varphi_n F = F_n$ so that $F_n \in H(A,A)$. It suffices to show $\sup \|F_n(A)\| < \infty$. Referring to (15) and (16) with some fixed $0 < s < 1$ and $\rho > v > \sigma$ for $x \in X$, $x^* \in X^*$ with $\|x\|, \|x^*\| \le 1$ we obtain the estimate for $1 \le t \le 2$:

$$\left|\left\langle M_{\pm}(t)x, x^*\right\rangle\right| \le \sum_{k\in\mathbb{Z}} \left|\left\langle F_n\left(2^{-k}t^{-1}e^{\pm iv}\right) g\left(2^k tA\right)x, g\left(2^k tA\right)^* x^*\right\rangle\right|$$

where $g(z) = \left(h_s^{\pm v}(z)\right)^{\frac{1}{2}}$. Suppose C is the U-bounded constant of $\left\{F(z) : z \in \sum_\rho\right\}$. Then

$$\left|\left\langle M_{\pm}(t)x, x^*\right\rangle\right| \le C \sup_{\varepsilon_k = \pm 1} \sup_N \left\| \sum_{|k|\le N} \varepsilon_k g\left(2^k tA\right)\right\|^2 .$$

Hence by Lemma (1.5) we have

$$\sup_n \|F_n(A)\| < \infty$$

Let us apply this to the case of two operators.

Theorem (1.10)[3]:

Suppose A, B are sectorial operators, such that A admits a $H^\infty\left(\sum_\sigma\right)$-calculus and $\omega(B) < \sigma'$. Suppose $f \in H^\infty\left(\sum_\rho \times \sum_{\sigma'}\right)$ where $\sigma < \rho < \pi$ is such that $\left\{f(\omega) : \omega \in \sum_\rho\right\}$ is contained in $H(B)$. Suppose further the set $\left\{f(\omega, B) : \omega \in \sum_\rho\right\}$ is U-bounded. Then $f \in H(A,B)$ (i.e. $f(A,B)$) is a bounded operator).

Proof:

We define $F(\omega) = f(\omega, B)$ and note that $F \in \left(\sum_\rho; A\right)$; this follows easily from the integral representation (4). Our conditions and Theorem (1.9) ensure that $F \in H(A,A)$. It is only necessary to check that this implies $f \in H(A,B)$ and of course $F(A) = f(A,B)$. But this follows directly from (4), (5) and the remarks thereafter.

Let us show by example that Theorem (1.10) is closed to the possible. Let B be a sectorial operator on X. Suppose $0 < \sigma < \pi$ and consider the space $L_2\left(\{-1,1\}^{\sum_\sigma}; X\right)$ where $\{-1,1\}^{\sum_\sigma}$ has the usual product measure.

Denote by ε_z the co-ordinate maps for $z \in \sum_\sigma$. Let Rad X denote the closed linear span of the functions $\left\{\varepsilon_z \otimes x : z \in \sum_\sigma, x \in X\right\}$. We define $B = I \otimes B$ on $L_2(X)$ and restrict it to the sub-space Rad X which is invariant. We define A on Rad X by

$$A\left(\sum_{z\in\sum_\sigma} \varepsilon_z x_z\right) = \sum_{z\in\sum_\sigma} z\varepsilon_z x_z$$

with domain consisting of all $\sum \varepsilon_z x_z \in L_2$ so that $\sum z\varepsilon_z x_z \in L_2$.

Clearly A has an $H^\infty\left(\sum_\sigma\right) -$ calculus and $f(A,B)$, for some $f \in H^\infty\left(\sum_\sigma \times \sum_{\sigma'}\right)$ with $\sigma' > \omega(B)$, is bounded if and only if the operators $f(z,B), z \in \sum_\sigma$, exist in $B(X)$ and form a R-bounded set.

We now consider strengthening of the boundedness conditions in the definition of sectoriality. Let A be a sectorial operator and let $\omega(A)$ denote the infimum of all σ so that A is of type σ.

We will say that A is R-sectorial, (respectively WR-sectorial, respectively U-sectorial) if there exists $0<\sigma<\pi$ so that the family of operators $\{\lambda R(\lambda, A):|\arg\lambda|>\sigma\}$ is R-bounded (respectively WR-bounded, respectively U-bounded). We then define $\omega_R(A)$, (respectively $\omega_{WR}(A)$, respectively $\omega_U(A)$ to be the infimum of all such σ. We will say A is $H^\infty-$sectorial (respectively, $RH^\infty-$sectorial, respectively $WFH^\infty-$sectorial) if there exists a $0<\sigma<\pi$ so that A admits an $H^\infty(\Sigma_\sigma)$-calculus (respectively, such that the set $\left\{f(A):\|f\|_{H^\infty(\Sigma_\sigma)}\le 1\right\}$ is R-bounded, respectively such that the set $\left\{f(A):\|f\|_{H^\infty(\Sigma_\sigma)}\le 1\right\}$ is WR-bounded). The infimum of all such σ is denoted $\omega_H(A)$ (respectively $\omega_{RH}(A)$, respectively $\omega_{WRH}(A)$).

There are certain obvious and trivial relationships between these concepts. Clearly R-sectorial implies WR-sectorial implies U-sectorial and whenever these concepts are defined, $\omega_R(A)\ge\omega_{WR}(A)\ge\omega_U(A)$.

Similarly RH^∞-sectorial implies $WRH^\infty-$ sectorial implies $H^\infty-$sectorial and $\omega_{RH}(A)\ge\omega_{WRH}(A)\ \ge\omega_H(A)\ge\omega(A)$
We now turn to less trivial observations:

Proposition (1.11)[3]:

Suppose A is H^∞- sectorial and U-sectorial. Then $\omega_H(A)\le\omega_U(A)$.

Proof:

Let us assume that $\{\lambda R(\lambda, A):|\arg\lambda|\ge v\}$ is $U-$bounded with constant k where $v>\omega(A)$, and $\sigma>v$. We will show that A admits an $H^\infty(\Sigma_\sigma)-$calculus. We use Proposition (1.8). Fix some $0<s<1$. We can assume that there exists $\rho>\sigma$ so that $\sup\limits_N\sup\limits_{\varepsilon_k=\mp1}\sup\limits_{t>0}\left\|\sum\varepsilon_k h_s^{\pm\rho}\left(2^k tA\right)\right\|=M<\infty$ and so that A admits an $H^\infty(\Sigma_\tau)-$calculus for some $t<\rho$.

Now suppose $x\in X$ and $x^*\in X^*$. Then for only N and $\varepsilon_j=\pm1$ we have

$$\left|\left\langle\sum_{|k|\le N}\varepsilon_k h_s^v\left(2^k tA\right)x, x^*\right\rangle\right|\le M\|x\|\left\|x^*\right\|+\left|\left\langle\sum_{|k|\le N}\left(h_s^v\left(2^k tA\right)\right)-\left(h_s^\rho\left(2^k tA\right)\right)x, x^*\right\rangle\right|$$

By the resolvent equation,

$$h_s^v\left(2^k tA\right)-h_s^\rho\left(2^k tA\right)=\left(e^{i(\rho-v)}-1\right)2^{-k}t^{-1}e^{iv}R\left(2^{-k}t^{-1}e^{iv}, A\right)h_s^\rho\left(2^k tA\right)$$

Since A has an $H^\infty(\Sigma_\tau)-$calculus we can define $g(\tau)=\left(h_s^\rho(z)\right)^{1/2}$ and note that

$$\sup_N\sup_{\varepsilon_k=\pm1}\left\|\sum_{|k|\le N}\varepsilon_k g\left(2^k tA\right)\right\|\le C$$

Where C is independent of t. Thus, by the boundedness of $\{\lambda R(\lambda, A):\arg\lambda=v\}$

$$\sum_{|k|\le N}\left|\left\langle 2^{-k}t^{-1}R\left(2^{-k}t^{-1}e^{iv}, A\right)g\left(2^k tA\right)x, g\left(2^k tA\right)\right\rangle\right|\le kC^2\|x\|\left\|x^*\right\|$$

It follows that

$$\left|\left\langle\sum_{|k|\le N}\varepsilon_k h_s^v\left(2^k tA\right)x, x^*\right\rangle\right|\le\left(M+2kC^2\right)\|x\|\left\|x^*\right\|$$

and this gives

$$\sup_N \sup_{\varepsilon_k=\pm1} \sup_{t>0} \left\| \sum_{|k|\le N} \varepsilon_k h_s^{\nu}\left(2^k tA\right) \right\| \le M + 2kC^2 < \infty .$$

Combined with a similar estimate for $-\nu$ we obtain the result by using Proposition (1.8).

In order to study an analytic semi-group with generator $(-A)$ it is of particular interest to know that $\omega_H(A) \le \pi/2$. Therefore we use Proposition (1.11) to improve on a result of $X.T$. Duong.

Corollary (1.12)[3][2]:

Let $(-A)$ generate an analytic contractive and positive semi-group on $L_p(\Omega,\mu)$ for some $1 < p < \infty$. Then $\omega_H(A) < \pi/2$.

Proof:

It is shown that $\omega_H(A) < \pi$ and $\omega_R(A) < \pi/2$. Hence we can apply Proposition (5.1.11).

We remark that it is an open problem whether $\omega_H(A) = \omega(A)$ whenever A is $H^\infty-$sectorial. The next theorem gives some results in this direction.

Theorem (1.13)[3]:

Suppose A is $H^\infty-$sectorial operator on a Banach space X.

Then:

(i) If X has property (α) then A is $RH^\infty-$sectorial and $\omega_H(A) = \omega_{RH}(A) = \omega_R(A) = \omega_U(A)$.

(ii) If X has property (A) then A is $WRH^\infty-$sectorial and $\omega_H(A) = \omega_{WRH}(A) = \omega_{WR}(A) = \omega_U(A)$

(iii) If X has property (Δ) then A is R-sectorial and $\omega_H(A) = \omega_R(A) = \omega_U(A)$.

Proof:

(i) Assume that A admits an $H^\infty(\Sigma_\sigma)-$calculus. Suppose $\sigma < \nu < \pi$. Suppose $0 < s < 1$ and let $g_\pm(z) = \left(h_s^{\pm\nu}(z)\right)^{\frac{1}{2}}$. We then can argue by Lemma (1.5) that

$$\sup_N \sup_{\varepsilon_k=\pm1} \left\| \sum_{k=-N}^{N} \varepsilon_k g_\pm\left(2^k tA\right) \right\| \le M < \infty$$

independent of t. Hence by Theorem (1.3) the family $\left\{ \sum_{|k|\le N} \alpha_k h_s^{\pm\nu}\left(2^k tA\right) \right\}$ is R-bounded with constant bounded independent of t. Now by (15) and (16) it follows that if $\sigma' > \nu$ then $\left\{ f(A) : \|f\|_{H^\infty(\Sigma_{\sigma'})} \le 1 \right\}$ is Rademacher-bounded. Indeed for $f_k \in H_0^\infty(\Sigma_{\sigma'})$ and $x_k \in X$ for $1 \le k \le n$ we have

$$\left(E\left\| \sum_{k=1}^{n} \varepsilon_k f_k(A) x_k \right\|^2 \right)^{1/2} \le 4 \max_{\pm} \sup_{t>0} \sup_{N\in\mathbb{Z}} \left(E\left\| \sum_{k=1}^{n} f_n\left(e^{\pm i\nu} t^{-1} 2^{-j}\right) h_s^{\pm i\nu}\left(2^{-j} tA\right) x_k \right\| \right)$$

It follows that $\omega_{RH}(A) = \omega_H(A)$. Now clearly $\omega_U(A) \le \omega_R(A) \le \omega_{RH}(A)$ and so (i) follows from Proposition (1.11).

(i) Is very similar.

(ii) Here we use Lemma (1.4). Suppose A admits an $H^{\infty}(\Sigma_{\sigma})$ - calculus and suppose $\sigma' > v > \sigma$. We show that the sequence $\{2^k tR(2^k te^{\pm iv}) : k \in Z\}$ is Rademacher-bounded with constant independent of t. To do this we note that if $N_1 > N_2$.

$$2^{N_1} tR\left(2^{N_1} te^{iv}, A\right) - 2^{N_2} tR\left(2^{N_2} te^{iv}, A\right)$$
$$= -\sum_{j=N_2+1}^{N_1} t2^{j-1} AR\left(2^j tR^{iv}, A\right) R\left(2^{j-1} te^{iv}, A\right)$$

Let $k(z) = z\left(e^{iv} - z\right)^{-1}\left(e^{iv} - 2z\right)^{-1}$. Let $u(z) = \left(k(z)\right)^{\frac{1}{2}} \in H^{\infty}(\Sigma_{\sigma})$. We observe that

$$\sup_{N_1 > N_2} \sup_{\varepsilon_j = \pm 1} \left\| \sum_{j=N_2+1}^{N_1} \varepsilon_j u\left(2^{-j} t^{-1} A\right) \right\| \le M < \infty .$$

independent of t by lemma (1.5). Applying Theorem (1.3) yields that

$$\left\{ \sum_{j=N_2+1}^{N_1} k\left(2^{-j} t^{-1} A\right) : N_1 > N_2 \right\}$$

is Rademacher-bounded with constant independent of t. But this implies that

$$\left\{ 2^{N_1} tR\left(2^{N_1} te^{iv}, A\right) - 2^{N_2 tR}\left(2^{N_2} te^{iv}, A\right) : N_1 > N_2 \right\}$$

is also Rademacher-bounded with constant independent of t. But this implies that $\left\{ 2^{N_1} tR\left(2^{N_1} te^{iv}, A\right) - 2^{N_2} tR\left(2^{N_2} te^{iv}, A\right) : N_1 > N_2 \right\}$ is also Rademacher-bounded with constan t independent of t and hence (taking limits) so is $\left\{ 2^n tR\left(2^n te^{iv}, A\right) : n \in N \right\}$. A similar argument for $-v$ and an application of Lemma (1.4) shows that $\omega_R(A) \le v$. Hence $\omega_R(A) \le \omega_H(A)$. The proof is complete as in (i).

As a corollary to the proof of Theorem (1.13) we obtain some additional information on the operator-valued calculus. Considered in Theorem (1.3).

Corollary (1.14)[3]:

Assume that X has property (α) and let $F \subset L(X)$ be an R-bounded set. If A is $H^{\infty}-$ sectorial then for any $\sigma > \omega_H(A)$ the set $\left\{ F(A) : F \in H^{\infty}(\Sigma_{\sigma}, A), F(\zeta) \in F \ \forall \zeta \in \Sigma_{\sigma} \right\}$ is R-bounded.

Proof:

Adapt the proof of Theorem (1.13) (i) using the fact that the set

$$\left\{ \sum T_k h_s^{\pm v}\left(2^k tA\right) : T_k \in F \cap A \right\}$$

Is R-bounded, again by Theorem (1.3).

Theorem (1.15)[3][1]:

Suppose A and B are $H^{\infty}-$sectorial operators such that B is $WRH^{\infty}-$sectorial. Then for any $\sigma > \omega_H(A)$ and $\sigma' > \omega_{WRH}(B)$ the pair (A, B) has a joint $H^{\infty}(\Sigma_{\sigma} \times \Sigma_{\sigma'})$-calculus.