Existence and Uniqueness of a Solution of Differential Equations with Constant Periodic Operator Coefficients

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Abstract

In this paper we establish a theorem that proves the existence and uniqueness of the solution of the equation $L_p^2 u(t) = f(t)$ *under some conditions on the unbounded operator. This operator has a domain and a range belonging to Hilbert's Space. There are no general methods which may allow to establish if a periodic solution exists for some specific system of differential equations or not, and that is the reason that our problem is not trivial and has an important value in the theory of differential equation with operator periodic coefficients.*

Keywords: Periodic Solutions, Operator Coefficients,

Introduction

Searching for periodic solutions for differential equations is not trivial. The main reason being that there are no general methods which may allow to establish if a periodic solution exists for some specific system of differential equations or not. Different methods and concepts should be inspected to find the best option but globally many of these methods are related to the perturbation theory [11].

In applied mathematics and physics, second order differential equations or the equivalent system of two first order equations have a great importance [2].

Many problems in physics and engineering lead to a system of linear differential equations with periodic coefficients. Lyapinov and Poincaré, who investigated the stability of periodic motions which are described by nonlinear differential equations transformed the centroid problem into a system of linear differential equations with periodic coefficients [5].

In the last years many results were achieved in the mathematical theory of differential equations with periodic coefficients, see [1, 3, 6, 7, 8, 12, 13, 16].

Piao [13] investigated the existence and uniqueness of periodic and almost periodic solution of the differential equation with reflection of argument. The relationship between modules of forced term and solution of the equation is considered

Benkhalti and Ezzinbi [1] studied the periodic solutions for some partial functional differential equations .

Li and Zhang [8] dealt with the existence of positive T-periodic solutions for the damped differential equation $\ddot{x} = p(t)\dot{x} = q(t)x = f(x,t)c(t)$, where p, q, $c \in L^1(\mathbb{R})$ are T-periodic functions and $f \in Car\left(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}\right)$ is T-periodic in the first variable. According to Li's work, this proves that a weak repulsive singularity enables the achievement of new existence criteria through a basic application of Schauder's fixed point Theorem.

Huseynov [6] investigated nonlinear second order differential equations subject to linear impulse conditions and periodic boundary conditions. Sign properties of an associated Green's function are exploited to get existence results for positive solutions of the nonlinear boundary value problem with impulse. The results obtained yield periodic positive solutions of the corresponding periodic impulsive nonlinear differential equation on the whole real axis.

Lillo [9] indicates the extension of some of the results of Hahn [4] for the Green's function to equations of the form considered by Shimanov [14] for periodic differential difference equations. He also indicates the relation of this Green's function to the representation problem. The results of Zverkin [17] for the case of a scalar equation where the lags are multiples of the period are studied. Convergence result for the series associated with Green's function is established. This result along with those of Zverkin and Lillo [10] indicate a kind of "harmonic resonance" which occur in these equations. Nieto [12] obtained under suitable conditions, the Green's function to express the unique solution for a second-order functional differential equation with periodic boundary conditions and functional dependence given by a piecewise constant function. This expression is given in terms of the solutions for certain associated problems. The sign of the solution is determined taking into account the sign of that Green's function.

Zhang and Wang [16] establish the existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order ordinary differential equations . The arguments are based only upon the positivity of the Green's functions and the Krasnosel'skii fixed point theorem. They apply their results to a problem coming from the theory of nonlinear elasticity. Cabado and Cid [3] give a L^p -criterium for the positiveness of the Green's function of the periodic boundary value problem: $x'' + a(t)x = 0$, $x(0) = x(T)$, $x'(0) = x'(T)$ with an indefinite potential a(t). Moreover, they prove that such Green's function is negative provided $a(t)$ belongs to the image of a suitable periodic Ricatti type operator.

Theoretical Frame

Consider the second order equation:

$$L_p^2 u(t) = D_t^2 u(t) - \sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj} M_{hkj} D_t^k u(t) = f(t) \tag{1}$$

with unbounded operator A_{kj} , the domain belongs to a Hilbert's space X and the range to a Hilbert's space Y , $X \subset Y$, $\|\bullet\|_X \geq \|\bullet\|_Y$, h_{kj} = constant, $M_{h_{kj}} u(t) = u(t - h_{kj})$, $D_t^k = \frac{1}{i^k}\frac{d^k}{dt^k}$.

Hereafter, we assume that $h_{00} = h_{10} = 0$. This condition allow us to include in (1) the classical solution without deviating argument. It is assumed that:

$$\begin{aligned} &f(t) \text{ is } \omega - \text{periodic function},\\ &A_{kj} : Y \to Y \text{ is closed operators},\\ &A_{kj} : X \to Y \text{ is bounded operators}. \end{aligned}$$

The existence of ω - periodic solutions of (1) is the main question in this work. For this, we consider the complete orthogonal system of functions $\left\{ e^{\frac{2i\pi nt}{\omega}}, n = 0, \pm 1, \ldots \right\}$ in Hilbert's space $L_2(0,\omega)$ and we expand the function $f(t)$ in a series with this system, *i.e.*

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{\frac{2i\pi nt}{\omega}} \tag{2}$$

Multiplying both sides of (2) by $e^{\frac{-2i\pi nt}{\omega}}$ and integrating the resulting equation from 0 to ω, we get

$$\int_0^{\omega} f(t)e^{\frac{-2i\pi nt}{\omega}}dt = \sum_{n=-\infty}^{\infty} f_n \int_0^{\omega} e^{\frac{2i\pi(n-R)t}{\omega}}dt$$

$$= \sum_{n=-\infty}^{\infty} f_n \begin{cases} \int_0^{\omega} dt, & n = k \\ 0, & n \neq k \end{cases}$$

$$= \omega f_k.$$

This implies that

$$f_k = \frac{1}{\omega}\int_0^{\omega} f(t)e^{\frac{-2i\pi kt}{\omega}}dt,$$

Thus one can consider the expansion $f(t) : \sum_{n=-\infty}^{\infty} f_n e^{\frac{2i\pi nt}{\omega}}, \quad f_n = \frac{1}{\omega}\int_0^{p} f(s)e^{\frac{-2i\pi ns}{\omega}}ds.$

We will seek a periodic solution of (1) in the form of a Fourier series

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{\frac{2i\pi nt}{\omega}}, t \in \mathrm{R}.$$

Substituting $u(t), f(t), \quad D_t^k u(t) = \sum_{n=-\infty}^{\infty} u_n \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{2i\pi nt}{\omega}}, \quad k = 0,1,$ and

$$M_{h_{kj}} D_t^k u(t) = \sum_{n=-\infty}^{\infty} u_n \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi nh_{kj}}{\omega}} e^{\frac{2i\pi nt}{\omega}}$$

into (1), we get

$$L_p^2 u(t) = \sum_{n=-\infty}^{\infty} u_n \left(\frac{2\pi n}{\omega}\right)^2 e^{\frac{2i\pi nt}{\omega}} -$$

$$\sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} \sum_{n=-\infty}^{\infty} u_n \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{2i\pi n(t-h_{kj})}{\omega}}$$

$$= f(t),$$

$$L_p^2 u(t) = \sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} \left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi nh_{kj}}{\omega}}\right] u_n$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{\frac{2i\pi nt}{\omega}},$$

equating the coefficients with the same powers of the exponential functions, we get

$$\left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi nh_{kj}}{\omega}}\right] u_n = f_n, \tag{3}$$

Assuming that the following condition holds:

$$\frac{2\pi n}{\omega} \in \rho(A_p) - \text{resolventsetoftheoperator} \tag{4}$$

$$A_p = \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} M_{h_{kj}} D_t^k : X \to Y, n = 0, \pm 1, \ldots, \tag{5}$$

which means that the spectrum of the operator A_p does not contain the points of real axis $\frac{2\pi n}{\omega}$, $n = 0, \pm 1, \ldots$, and from (3) we find

$$u_n = \left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi n h_{kj}}{\omega}} \right]^{-1} f_n. \tag{6}$$

If the equation

$$\left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi n h_{kj}}{\omega}} \right]^{-1} \varphi_n = 0. \tag{7}$$

has a nontrivial solution $\varphi_0 \in X$, then the numbers $\frac{2\pi n}{\omega}$ belong to the spectrum of the operator

$$A_p = \sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj} M_{h_{kj}} D_t^k.$$

Introducing the notation

$$R_n = \left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi n h_{kj}}{\omega}} \right]^{-1},$$

equation (6) can be written in the form

$$u_n = R_n f_n, \tag{8}$$

which gives

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{\frac{2i\pi n t}{\omega}}$$
$$= \sum_{n=-\infty}^{\infty} R_n f_n e^{\frac{2i\pi n t}{\omega}}, \tag{9}$$

where the resolvant operator $R_n : Y \to X$ depends on the parameter n. By Virtue of the enclosure $X \subset Y$, we can consider the operator $R_n : Y \to Y$. Introducing in the right side of (9) the value of f_n, we get

$$u(t) = \int_0^{\omega} G(t-s) f(s) ds, \tag{10}$$

where

$$G(t-s) = \frac{1}{\omega} \sum_{n=-\infty}^{\infty} \left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi n h_{kj}}{\omega}} \right]^{-1} e^{\frac{2i\pi n(t-s)}{\omega}}, \tag{11}$$

or

$$G(t-s) = \frac{1}{\omega} \sum_{n=-\infty}^{\infty} R_n e^{\frac{2i\pi n(t-s)}{\omega}}.$$

We consider the function $\xi(t)=\frac{t}{2}(\frac{t}{\omega}-1),\quad 0<t<\omega.$ Since,

$$\xi_n=\frac{1}{\omega}\int_0^{\omega}\left[\frac{t}{2}\left(\frac{t}{\omega}-1\right)\right]e^{\frac{-2i\pi nt}{\omega}}dt$$

$$=\frac{1}{2\omega^2}\int_0^{\omega}t^2e^{\frac{-2i\pi nt}{\omega}}dt-\frac{1}{2\omega}\int_0^{\omega}te^{\frac{-2i\pi nt}{\omega}}dt$$

$$=\frac{1}{2\omega^2}\left[t^2\frac{-\omega}{2i\pi n}e^{\frac{-2i\pi nt}{\omega}}\Bigg|_0^{\omega}+\int_0^{\omega}2t\frac{\omega}{2i\pi n}e^{\frac{-2i\pi nt}{\omega}}dt\right]$$

$$-\frac{1}{2\omega}\left[t\frac{-\omega}{2i\pi n}e^{\frac{-2i\pi nt}{\omega}}+\int_0^{\omega}\frac{\omega}{2i\pi n}e^{\frac{-2i\pi nt}{\omega}}dt\right]$$

$$=\frac{\omega}{(2\pi n)^2}$$

and then subtracting from both sides of (11) the ω - periodic function ξ (t) E, expressed in uniformly convergent series ω - periodic functions

$$\xi(t)E=\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}e^{\frac{2i\pi nt}{\omega}}E, \tag{12}$$

we get

$$G(t)-\xi(t)E=\frac{1}{\omega}\sum_{n=-\infty}^{\infty}R_ne^{\frac{2i\pi nt}{\omega}}-\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}e^{\frac{2i\pi nt}{\omega}}E$$

$$=\frac{-1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}+\sum_{n\neq 0}\left(\frac{1}{\omega}R_n-\frac{\omega}{(2\pi n)^2}E\right)e^{\frac{2i\pi nt}{\omega}}$$

$$G(t)=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}+\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left[\left(\frac{2\pi n}{\omega}\right)^2R_n-E\right]e^{\frac{2i\pi nt}{\omega}}],$$

adding and subtracting inside the square brackets of the operator

$$\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi nh_{kj}}{\omega}}, \tag{13}$$

we get

$$G(t)=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}$$

$$+\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\left[\left(\frac{2\pi n}{\omega}\right)^2E-\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi nh_{kj}}{\omega}}\right.\right.$$

$$\left.\left.+\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi nh_{kj}}{\omega}}\right]R_n-E\right)e^{\frac{2i\pi nt}{\omega}}$$

$$=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}$$

$$+\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\left[\left(\frac{2\pi n}{\omega}\right)^2 E-\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi n h_{kj}}{\omega}}\right]R_n\right.$$

$$\left.+\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi n h_{kj}}{\omega}}R_n-E\right)e^{\frac{2i\pi n t}{\omega}}$$

$$=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}+\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k$$

$$e^{\frac{-2i\pi n h_{kj}}{\omega}}e^{\frac{2i\pi n t}{\omega}}R_n, \tag{14}$$

or

$$G(t)=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}$$

$$\left(\frac{2\pi n}{\omega}\right)^k A_{kj}R_n e^{\frac{2i\pi n\left(t-h_{kj}\right)}{\omega}}. \tag{15}$$

For the series $\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\frac{2\pi n}{\omega}\right)^k R_n e^{\frac{2i\pi n\left(t-h_{kj}\right)}{\omega}}$ the majorant will be the series $\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\frac{2\pi n}{\omega}\right)^k \left\|A_{kj}R_n\right\|_Y$, where

$$\left(\frac{2\pi n}{\omega}\right)^k\left\|A_{kj}R_n\right\|_Y=\left\|A_{kj}\left(\frac{2\pi}{\omega}\right)^k n^k R_n\right\|_Y\leq$$

$$\begin{cases} c\left\|n^k R_n\right\|_X, & k=0,1\\ c\left\|n^k R_n\right\|_Y, & k=2\end{cases}$$

Hence, if we require that the conditions

$$\left\|n^k R_n\right\|_X=O(1),\, k=0,1,$$

$$\left\|n^k R_n\right\|_Y=O(1), \tag{16}$$

which are equivalent to $\left\|R_n\right\|_X=O(\frac{1}{\left|n^k\right|})$, $k=0,1$ and $\left\|R_n\right\|_Y=O(\frac{1}{\left|n^2\right|})$ must be held, then the series in (14) converges absolutely and uniformly and its sum is a continuous and periodic function.

Theorem

If for any n there exists R_n.

$$\sum_{n=-\infty}^{\infty}\left\|nR_n\right\|_X=O(1),$$

$$\left\|n^2 R_n\right\|_Y=O(1),\, n\to\infty \tag{17}$$

then

$$L_p^2 : X_{(0,\omega)}^{2,0} \to Y_{(0,\omega)}^{0,0} \tag{18}$$

is continuously invertible.

Proof

We use Parseval's equality which expresses the norm square of an element in a space with scalar product through the norm square of Fourier coefficients of this element by some orthogonal system of elements. In our cas, the equality:

$$\sum_{n=-\infty}^{\infty} \| f_n \|_Y^2 = \frac{1}{\omega} \int_0^{\omega} \| f_n \|_Y^2 \, dt \tag{19}$$

$$\sum_{n=-\infty}^{\infty} \| f_n \|_Y^2 = \frac{1}{\omega} \int_0^{\omega} \| f_n \|_Y^2 \, dt \tag{20}$$

and theorem conditions on $f(t)$ implies the existence of $f_n \in Y, n = 0, \pm 1, \ldots$

By virtue of theorem conditions on R_n there exists $U_n = R_n f_n \in X$.

We consider the series

$$\sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} R_n f_n,$$

$$\sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} \frac{2i\pi n}{\omega} R_n f_n, \tag{21}$$

$$\sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} \frac{2\pi n}{\omega}^2 n^2 R_n f_n,$$

where we get the last two series from the first one by formal differentiation. By virtue of theorem conditions on R_n and by Parseval's equality we get

$$a) \quad \int_0^{\omega} \left\| \sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} R_n f_n \right\|_X^2 dt = \omega \sum_{n=-\infty}^{\infty} \| R_n f_n \|_X^2 \le \omega \sum_{n=-\infty}^{\infty} \| R_n \|_X^2 \| f_n \|_Y^2 \le c \int_0^{\omega} \| f(t) \|_Y^2,$$

$$b) \quad \int_0^{\omega} \left\| \sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} \frac{2i\pi}{\omega} n R_n f_n \right\|_X^2 dt = \omega (\frac{2\pi}{\omega}^2) \sum_{n=-\infty}^{\infty} \| n R_n f_n \|_X^2 \le c_1 \int_0^{\omega} \| f(t) \|_Y^2,$$

$$c) \quad \int_0^{\omega} \left\| \sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} i^2 \frac{2\pi}{\omega}^2 n^2 R_n f_n \right\|_X^2 dt = \omega (\frac{2\pi}{\omega}^4) \sum_{n=-\infty}^{\infty} \| n^2 R_n f_n \|_X^2 \le c_2 \int_0^{\omega} \| f(t) \|_Y^2, \tag{22}$$

From the derived inequalities a, b and c implies the existence of the functions: $u(t)$, $u'(t)$ and $u''(t)$, through which we denote the convergent series (21) , and also the estimations

$$\int_0^{\omega}\|u(t)\|_X^2\,dt \le c\int_0^{\omega}\|f(t)\|_Y^2, \tag{23}$$

$$\int_0^{\omega}\|u'(t)\|_X^2\,dt \le c_1\int_0^{\omega}\|f(t)\|_Y^2, \tag{24}$$

$$\int_0^{\omega}\|u''(t)\|_X^2\,dt \le c_2\int_0^{\omega}\|f(t)\|_Y^2, \tag{25}$$

From which implies that $u(t)\in X^{2,0}_{(0,\omega)}$. Now, by direct substitution, we verify that the function

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{\frac{2i\pi nt}{\omega}} = \sum_{n=-\infty}^{\infty} R_n f_n e^{\frac{2i\pi nt}{\omega}}$$

is ω-periodic solution of (1), since each term of this series has the period ω. In fact:

$$L_p^2 u(t) = D_t^2 u(t) - \sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj} S_{hk_j} D_t^k u(t)$$

$$= \sum_{n=-\infty}^{\infty} R_n f_n \left(\frac{2\pi n}{\omega}\right)^2 e^{\frac{2i\pi nt}{\omega}} -$$

$$\sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj} \sum_{n=-\infty}^{\infty} R_n f_n \cdot \left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi nt}{\omega}h_{kj}}) e^{\frac{2i\pi nt}{\omega}} = \sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} \left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi nt}{\omega}h_{kj}}\right] R_n f_n$$

$$= \sum_{n=-\infty}^{\infty} R_n^{-1} R_n f_n = \sum_{n=-\infty}^{\infty} f_n e^{\frac{2i\pi nt}{\omega}} = f(t)$$

Uniqueness

If we allow the existence of one more ω-periodic solution $u_1(t)$ of (1), then

$U(t) = u(t) - u_1(t)$ is also a ω-periodic solution of (1). Hence, from

$U(t)\in X^{2,0}_{(0,\omega)}$ by virtue of completeness of the system of functions: $e^{\frac{2i\pi nt}{\omega}}, n = 0, \pm 1, \ldots$ in $L_2(0,\omega)$, implies the equality $U_n = R_n \cdot 0 = 0,\ n = 0, \pm 1, \ldots$, for Fourier coefficients of the functions $U(t)$

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