The Sub-Optimal Effect of the Simplex Solution Method in Resource Allocation Using the Linear Programming Model

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Abstract

It is evident that linear programming model remains the most potent mathematical tool for the efficient allocation of scarce operational resources of an organization. Whilst projecting the graphical method as the easiest solution approach to linear programming where only two constraining factors are involved, the more complicated simplex method offers the freedom to solve problems involving more than two operational factors. A recent analysis of real life operational activities of a small manufacturing company using both methods to find the best mix for combining constraining resources for optimal performance, however, revealed that the simplex method though may be effective in dealing with multiple constraining factors, it does so at the price of sub-optimal decision solution truncating vital non-linear information ordinarily accounted for by other analytical methods. This paper suggests a modified approach to using the simplex method in solving business related linear programming problems to ensure optimal decision in resource allocation.

Keywords: Resources, Constrains, Linear, Production, Limiting Factors, Simplex

Introduction

In the world of business what separates the men from the boys is the ability to maximize the gains from the use of the available scarce operating resources available to the members of each group. Economists have long identified scarcity of operating resources as the main obstacle towards the attainment of the true satisfaction of human wants which unfortunately are too numerous and unsteady. One of the major problems facing producers of goods and services in a modern economy is how to cope with customers' demands, changes in consumers' tastes and meeting the ever changing preferences of an equally unsteady global shift in population age grouping.

To worsen the ever growing managerial dilemma of the modern entrepreneur in the area of resource management and efficient production, the advent of the internet and modern high-tech communication devices which inadvertently molded the world into a global village made competition and customers' demands more complex, thereby leaving only the very best of businesses which can cope with complex decisions at electronic speed to stay on. Survival competition is not only in terms of getting customers' attention, it emanates and even more pronounced at the point of sourcing for the scarce production resources which only those with the best of decision support tools can access.

Objectives of the Study

The primary aim of this paper is to highlight some noted hidden defects of the simplex method of solving linear programming models which are used in the allocation and re-allocation of scarce operational resources where more than two of such resources are involved and cannot be handled with the use of graph. The specific objective being to suggest a modified approach to using the simplex method in solving business related linear programming problems to ensure ***true*** optimal decision in resource allocation.

Methodology

The method adopted for this investigation was to analyze the resource allocation element in operational decisions of a small manufacturing company for the past six months using both the graphical method and the simplex method. The result was then further subjected to statistical tests using ANOVA cum 't' statistics and theoretical analysis with references to standard operations research and quantitative technique texts on the subject.

Literature Review

In the words of Lucey (2006), allocation problems are concerned with the utilization of limited resources to best advantage; and linear programming is one of those techniques used in tackling this. In their own contribution, Hillier and Lieberman (2004) stated that the development of linear programming was ranked among the most important scientific advances of the mid-20th century due its extraordinary impact since 1950. They further opined that linear programming is a standard tool that has saved millions of dollars for most companies and businesses in the various industrialized countries of the world.

Linear programming has equally found tremendous patronage by computer programmers who found it easy to be used for complex scientific computations (Sogunro and Adekanye, 2009). The most common type of application of the linear programming tool involves the general problem of allocating limited resources among competing needs or activities with the single purpose of attaining optimality in decision making (Hillier and Lieberman, 2004; Sogunro and Adekanye, 2009; Stafford, 1981; Lapin, 2007; Keyeke, 2006). Hillier and Lieberman (2004) went further to assert that any problem whose mathematical model fits the very general format for the linear programming model can be solved with linear programming technique. However, a major factor that influences the classification of a problem as a linear programming one is the ability to show that all relationships forming that problem have the required linear property (Lucey, 2006).

Having noted the usefulness of linear programming, one will be definitely tempted to look at its methodology. Apart from the requirement of linearity, the linear programming model must be capable of being formulated into *the objective function* and *the limitation functions.* There are two major methods of handling and solving linear programming problem – through the use of an ***x-y graph*** or by the use of the optimization method called the ***simplex*** method (Hillier and Lieberman, 2004; Lucey, 2006; Stafford, 1981). But the choice of any one method will depend on the number of ***decision variables*** involved. Lucey (2006) and Hillier and Lieberman (2004) agreed that a graphical solution is only possible when the model involves not more than two decision variables. The reason behind this position is that a graph can only be identified with two dimensions called axis x and y; and each of these axis has its positive and negative extensions in two opposite directions, thus covering all the four available points on the graph, thereby giving no chance for a three or more dimensional problems.

Here, we define a decision variable as ***a unit of product or cost element sharing the limited resources available to an operation with other units or elements*****.** These limited resources are the sources of the constrains that limits the expansion or contraction of any of the decision variables beyond certain perceivable points. The decision variables are the object and main focus of the objective functions of the linear programming model upon which the limited/constraining resources are optimized.

Illustrative Case Study

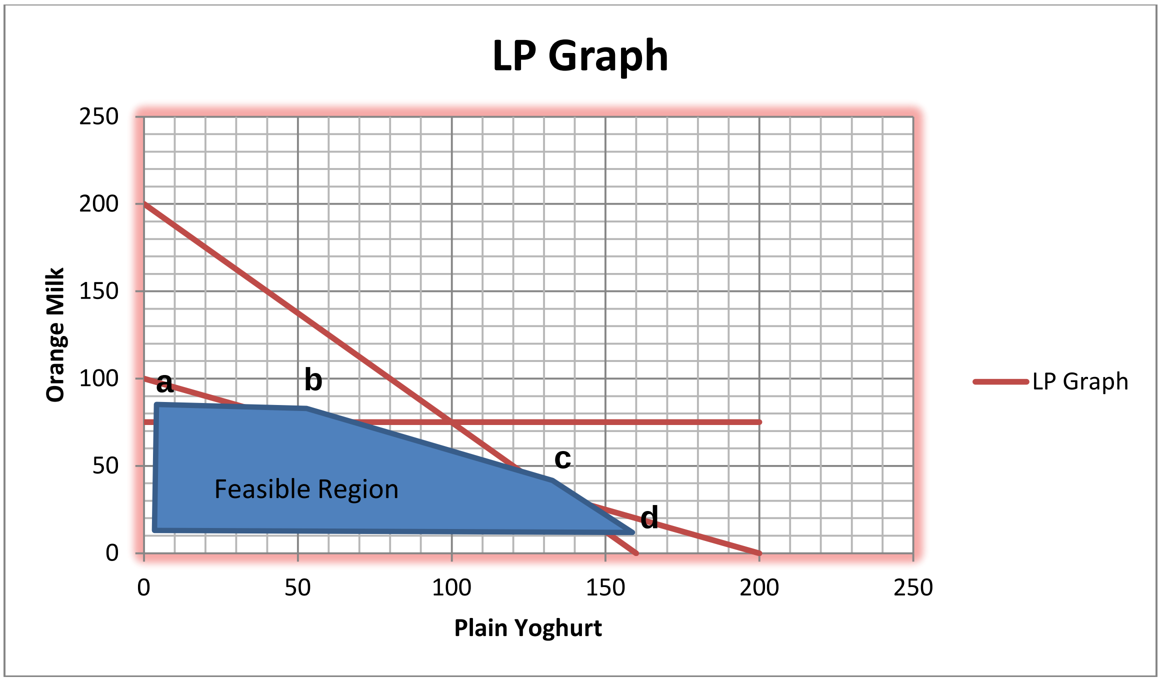
Serengeti Beverages (SB Ltd)

SB Ltd manufactures two related products *Plain Yoghurt* and *Orange Milk Drink*. Plain yoghurt sells for N60 a bottle (N1,440 per carton of 24) whilst orange milk drink sells for N70 a bottle (N1,680 per carton of 24). The contribution margins for the products are N400 and N500 per carton for yoghurt and milk respectively. Each product passes through almost similar production processes but with little differences in pasteurization and emulsification. The production inputs for the products are sizeable and can be easily obtained but there are two important inputs which are limited in supply – labour hours and a *special material*. A carton of plain yoghurt requires 2 labour hours while a carton of Orange Milk requires 4 but SB Ltd can only afford 40 men working 10 hours a day. On the other hand, a carton of Plain Yoghurt requires 2 measures of the *special material*, while that of Orange Milk requires only 1.6 measures of it but the supplier is only willing to supply a maximum of 320 measures per day. Presently, the major distributor has the capacity to sell only 75 cartons of orange milk a day and would not want to stock more than its daily marketing ability for fear of incurring losses on possible damages due to the perishable nature of the drink. On the basis of the above information, we shall calculate the best mix that will maximize the total contribution of SB Ltd., first using the graphical method and then using the simplex method.

Using ***p*** to represent plain yoghurt and ***m*** to represent orange milk, the objective function is:

Maximize **Z:** $400p + 500m$
Subject to:
$2p + 4m \leq 400$ 'labour hour constraints
$2p + 1.6m \leq 320$ 'special material constraint
$m \leq 75$ 'distribution constraint
Where ***p*** and ***m*** cannot be less than zero.
Note: 400 hours = 40 men multiplied by 10 hours

Figure 1: Graphical Solution



Source: Enyi, E. P. July 2012

The feasible region from the above table is bounded by the area marked with a, b, c, and d vertices.

Table 1: Evaluating the Graph

Point	Plain Yoghurt N400	Orange Milk N500	Total Contribution	Rank
a	0 x 400	75 x 500	37,500	4
b	50 x 400	75 x 500	57,500	3
c	135 x 400	32 x 500	70,000	**1** √
d	160 x 400	0 x 500	64,400	2

From the above evaluation, it is being suggested that the best daily production mix that will maximize the use of the two scarce resources is to produce 135 cartons of ***plain yoghurt*** and 32 cartons of ***orange milk***. This will give the maximum feasible contribution of N70,000 per day.

Simplex Solution

If simplex algorithm is suitable for the solution of a model with more than two decision variables, then it should be able to handle a model with two variables more perfectly. Let's see whether this is the case.

Table 2: The initial simplex tableau for the above business model:

Solution Variable	*Decision Variables* *P*	*M*	*Solution Slacks* x_1	x_2	x_3	*Solution Quantity*
x_1	2	4	1	0	0	400
x_2	2	1.6	0	1	0	320
x_3	0	1	0	0	1	75
Z	400	500	0	0	0	0

Table 3: The first iteration tableau using the maximum objective is:

Solution Variable	*Decision Variables* *P*	*M*	*Solution Slacks* x_1	x_2	x_3	*Solution Quantity*
x_1	2	0	1	0	-4	100
x_2	2	0	0	1	-1.6	200
x_3	0	1	0	0	1	75
Z	400	0	0	0	-500	-37,500

The value at the bottom of the *Solution Quantity* column (Z row) is the same value with point ***a*** on the graph evaluation table above. This is an indication that we are in the right direction but since the above zero value in the ***p*** column Z row is informing us that we are yet to reach the optimal solution, we have to stretch our algorithmic analysis further to be on the safe side.

Table 4: The second and final iteration tableau is presented below:

Solution Variable	*Decision Variables* *P*	*M*	*Solution Slacks* x_1	x_2	x_3	*Solution Quantity*
x_1	1	0	½	0	-2	50
x_2	0	0	-1	1	2.4	100
x_3	0	1	0	0	1	75
Z	0	0	-200	0	300	-57,500

Hillier and Lieberman (2004:131) states that the *simplex* method automatically stops after one optimal feasible solution is reached. Since there is no positive values on the Z row within the ***p*** and ***m*** columns, we can conclude that the supposedly optimal decision point is reached. Meaning that SB Ltd should produce 50 cartons of *plain yoghurt* and 75 cartons of *orange milk* giving a combined contribution margin of N57,500 per day. The reason for this apparent misleading information and other lessons to be learned from the above shall be given at the discussion segment.

To enable more elaborate analytical discussion, we shall take another case from an illustration in Lucey (2006:314-322) thus:

> A company can produce three products, A, B, and C. The products yield contributions of £8, £5 and £10 respectively. The products use a machine which has 400 hours capacity in the next period. Each unit of the products uses 2, 3 and 1 hour respesctively of the machine's capacity. There are only 150 units available in the period of a special component which is used singly in products A and C. 200 kgs of a special alloy is available in the period. Product A uses 2 kgs per unit and product C uses 4 kgs per unit. There is an agreement with a trade association to produce no more than 50 units of product B in the period. The company wishes to find out the production plan which maximizes contribution.

For the purpose of our discussion, we shall not bother ourselves with such preliminaries as model formulation, setting up the initial simplex tableau and intermediate algorithmic iterations, rather we shall go straight to reproduce the initial and final solution tableau and the attendant interpretations.

Table 5: The initial simplex tableau

Solution	Products			Slack Variables				Solution
Variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**Quantity**
x_4	2	3	1	1	0	0	0	**400**
x_5	1	0	1	0	1	0	0	**150**
x_6	2	0	4	0	0	1	0	**200**
x_7	0	1	0	0	0	0	1	**50**
Z	**8**	**5**	**10**	**0**	**0**	**0**	**0**	**0**

Table 6: The final simplex tableau

Solution	Products			Slack Variables				Solution
Variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**Quantity**
x_4	0	0	-3	1	0	-1	-3	**50**
x_5	0	0	-1	0	1	-½	0	**50**
X_1	1	0	2	0	0	½	0	**100**
X_2	0	1	0	0	0	0	1	**50**
Z	**0**	**0**	**-6**	**0**	**0**	**-4**	**-5**	**-1,050**

The basic interpretation given to the Z row in table 6 above is that the optimum solution is to produce 100 units of Product A and 50 units of Product B with none for Product C to give a total contribution of £1,050. There are 50 unused machine hours and 50 unused components with this solution.

Now, supposing we decided to reject this "*optimal*" solution and instead use the matrix algebra (rather than the simplex method) to do our computations which are presented below:

Table 7: The initial matrix table

Products	**A**	**B**	**C**	**Constrain**
Machine Hour	2	3	1	400
S. Component	1	0	1	150
Special Alloy	2	0	4	200
Trade Agrmt	0	1	0	50

Table 8: Manipulating the first column of the matrix table

Products	**A**	**B**	**C**	**Constrain**
Machine Hour	1	3/2	1/2	200
S. Component	0	-3/2	1/2	-50
Special Alloy	0	-3	3	-200
Trade Agrmt	0	1	0	50

Table 9: Manipulating the second column of the matrix table

Products	**A**	**B**	**C**	**Constrain**
Machine Hour	1	0	1	150
S. Component	0	1	-1/3	100/3
Special Alloy	0	0	2	-100
Trade Agrmt	0	0	1/3	50/3

Now the third and final matrix manipulation table requires our intuition because there are two possible outcomes; each with a different prescription. But this is where the test of optimality lies. Recall that the final simplex method table (table 6) talked about unused 50 units each of ***machine hours*** and the ***special component*** respectively. Since Product C uses each of these resources singly and has no need for the special alloy which has been fully utilized, then, it dawns on us that the true optimal decision will only be reached if we can find a solution that can fully utilize the unused resources and accommodate Product C. Only one of the tables below meets this criterion.

Table 10a: Final matrix manipulation table (first option – ignore trade agreement)

Products	A	B	C	Constrain
Machine Hour	1	0	0	200
S. Component	0	1	0	50/3
Special Alloy	0	0	1	-50
Trade Agrmt	0	0	0	100/3

Table 10a which ignored the trade agreement constraint is giving a solution similar to the simplex method that the firm should produce: 200 units of Product A; 16.67 units of Product B, and 0 units of Product C; since the non-negativity condition prevents the firm from producing -50 units of Product C which was indicated in the above solution.

Table 10b: Final matrix manipulation table (second option – ignore special alloy)

Products	A	B	C	Constrain
Machine Hour	1	0	0	100
S. Component	0	1	0	50
Special Alloy	0	0	0	-200
Trade Agrmt	0	0	1	50

Table 10b which rightly ignored the special alloy constraint because it has no bearing on Product C is giving a solution that the firm should produce: 100 units of Product A; 50 units of Product B, and 50 units of Product C; giving a total contribution of £1,550 (100x8 + 50x5 + 50x10) rather than the £1,050 provided by the simplex solution.

Discussion

A comparative look at figure 1 with its evaluation table (table 1) and the simplex solution table (table 4) will show that the simplex solution is not truly optimal in the real sense of optimality. This is because table 1 being an evaluation of the graphical solution seems to say a different thing. Table 1 shows that the optimal solution lies at the point where total contribution from the two decision variables equals N70,000 as against N57,500 in table 4 by the simplex method applying the rule for optimality test as mentioned earlier.

However, a closer look at the **Z row** of the **slack variables** in table 4 indicates that something can still be done to improve the total contribution. But before then, let us see how the value of slacks in ***table 6*** was variously interpreted:

*In **table 6,** the value of -6 under column Product x_3 means that if any unit of Product C (x_3) was produced then overall contribution would fall by £6. The values of Z row under columns x_4, x_5, x_6 and x_7 are equally important as they are valuations of resources known as **shadow prices** or **shadow costs**. Their values are interpreted as follows:*

> *0 for x_4 and x_5 means that no further value can be gained by adding more to them;*
> *-4 for x_6 and -6 for x_7 means that overall contribution can be increased by the respective amount for any further addition to x_6 and x_7.*

Having seen how the values of slacks were variously interpreted for ***table 6***, let us go back to ***table 4***, the final solution simplex tableau for our case study model. Here, we can see that x_1 has a Z row value (or ***shadow price***) of -200 showing that the sum of N200 can be gained for every extra unit of labour hour added to production; while x_3 has a Z row value (or ***shadow price***) of +300 showing that N300 can be gained for every carton of *orange milk* reduced and the resources added to produce more *plain yoghurt*. But here, the question is: how much of *orange milk* should we reduce to achieve the optimality? To answer this question, we have to look at all the elements on the *offending column* and decide which one of them should become the pivot element. This will be achieved by dividing the *solution quantity* values by their corresponding elements on the offending column to choose the element with the lowest division value. In this case row x_2 produced the lowest positive division value. The new final simplex solution table on the basis of this is presented below:

Table 11: True Final Simplex Solution Tableau for Our Case Model

Solution Variable	*Decision Variables*		*Solution Slacks*			*Solution Quantity*
	P	***M***	x_1	x_2	x_3	
x_1	1	0	-0.9167	0.4167	0	133.33
x_2	0	0	-0.4167	0.4167	1	41.67
x_3	0	1	0.4167	-0.4167	0	33.33
Z	0	0	-75	-125	0	-70,000

A look at the Z row of ***table 11*** above reveals that the offending resource x_3 has been used to settle the more productive ones x_1 and x_2. Meaning that the labour hour and *special material* resources constraints are now fully engaged with the capability to add more to the overall contribution (N75 for every labour hour added and N125 for every measure of *special material* added). We can now see that the new optimal solution is similar to that of the graphical solution with N70,000 maximum contribution but with slightly differing prescription. While the graphical method prescribes the production of 135 cartons of *Plain Yoghurt* and 32 cartons of *Orange Milk*, the simplex method suggests the production of 133.33 cartons of *Plain Yoghurt* and 33.33 cartons of *Orange Milk* with both giving the same total contribution of N70,000 a day. Please note that this same result can be obtained using the matrix algebra method as well.

Back to ***table 6***, we were told that the optimal solution would not include Product C, and also that 50 units of machine hours and 50 components of the special alloy would remain unutilized. If the simplex method is all about efficient manipulation of the Linear Programming model, why does it leave room for wastages? Since Product C has a positive contribution margin of £10 (the highest of the three products), why exclude it when there are unused resources? Strictly speaking, excluding Product C will most probably add to the overall operating cost of the firm due mainly to low capacity utilization; thereby reducing the overall profit of the firm and negating the purpose for which linear programming was introduced. This is clearly evident from the results of the matrix manipulation of the same set of data on table 10b which revealed that the firm will be losing £500 (1550-1050) if the simplex solution method is used. To put the above observations to statistical proof, the study collected the operational activities data for SB Ltd over a period of six months as indicated on ***table 14*** (Appendix 1). The test was based on the need to proof/disproof the hypothesis that:

> H_0 = ***The Simplex Method of Solving Linear Programming Models Does Not Lead To Sub-Optimal Decisions in Business***

The test was conducted on the plant's ***Capacity Utilization*** and ***Return On Investment*** ratios using the one way ANOVA statistics supported further by the '***t***' statistics as presented below:

Table 12: *Capacity Utilization* Tests Table

ANOVA SUMMARY								
Group	**Count**	**Sum**	**Average**	**Variance**				
Graph	11	1094	99.45	0.87				
Simplex	14	1060	75.71	24.68				
Statistics								
Source	**SS**	**df**	**MS**	**F**	**Table**			
B/W Grps	75.71	1	3471.78	271.72	4.23			
Within Grp	99.45	26	12.78					
Total	175.17	27						
't' Test Values								
Group	**Count**	**Mean**	**SD**	**SE**	**SEM**	**df**	**'t'**	**Table**
Graph	11	99.45	0.93	1.141	1.52	23	15.56	2.069
Simplex	14	75.71	4.77	1.012				

From the above results, it can be seen that both the ANOVA ***F*** ratio and the '***t***' value are substantially outside the tabulated value; therefore, on ***capacity utilization*** criterion, we reject the null hypothesis and accept the alternate.

Table 13: *Return on Investment* Tests Table

ANOVA SUMMARY								
Group	**Count**	**Sum**	**Average**	**Variance**				
Graph	11	304.5	27.68	1.81				
Simplex	14	327	23.36	0.98				
Statistics								
Source	**SS**	**df**	**MS**	**F**	**Table**			
B/W Grps	27.68	1	115.2	82.54	4.23			
Within Grp	23.36	26	1.396					
Total	51.04	27						
't' Test Values								
Group	**Count**	**Mean**	**SD**	**SE**	**SEM**	**df**	**'t'**	**Table**
Graph	11	27.68	1.35	0.35	0.47	23	9.27	2.069
Simplex	14	23.36	0.99	0.31				

From the results on ***table 13***, it can be seen that both the ANOVA ***F*** ratio and the '***t***' value are substantially outside the tabulated value; therefore, on ***return on investment*** criterion, we reject the null hypothesis and accept the alternate.

Overall, the test results show that ***the Simplex Method of solving Linear Programming Models Leads significantly to Sub-Optimal Decisions in Business*** whether measured from the angle of capacity utilization or from the angle of profitability.

Conclusion and Recommendations

Linear Programming is no doubt a very good mathematical tool for operations research especially in the area of scarce resources allocation; its application must, however, be followed with great caution especially when it involves the use of the simplex method because it involves a lot of *out-of-the-ordinary* mathematical manipulations which requires special and scientifically informed interpretations to understand, with each figure giving a different meaning and direction. The calculation sequence is also not that simple because the duration and sequence of iteration is dependent on the intermediate values obtained during the computation process. Every model must be developed and solved along line the peculiarity of the underlying problem for which it was formulated. Personal innovations are sometimes required in order to take the solution to the true optimal level as a model can have more than one optimal solution which a non-modified simplex method might not be able to detect, as it is in the SB Ltd case study. Managers are advised not to base their decisions strictly on the results generated using the simplex solution method as there are tendencies that the intrinsic and interacting mathematical relationships involved in its manipulation may throw out some important element that could contribute meaningfully to full capacity utilization of available resources of an organization on the guise that such element does not form part of the optimal simplex solution. The good news is that there is a way around and this is based on the golden rule that "***when in doubt, consult the slack values for shadow prices***", or better still – use matrix algebra.

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Appendix 1

Table 14: Summary of Operational Activities for 6 Months

Month/Week	No. of DV Involved	No. of CF Involved	No. of Production Days/Week	Evaluation Method	% Capacity Utilization	% ROI Achieved
Jan/1	2	3	5	Graph	100	27
Jan/2	2	3	5	Graph	100	26.5
Jan/3	2	3	6	Graph	98	25
Jan/4	2	3	5	Graph	100	27
Feb/1	2	3	5	Graph	100	28
Feb/2	3	3	5	Simplex	75	22
Feb/3	3	3	5	Simplex	80	23
Feb/4	3	3	5	Simplex	78	22
Mar/1	3	3	5	Simplex	78	24
Mar/2	3	3	5	Simplex	80	23
Mar/3	3	3	5	Simplex	78	24
Mar/4	3	3	5	Simplex	80	25
Mar/5	3	3	5	Simplex	78	22.5
Apr/1	4	3	5	Simplex	65	24
Apr/2	4	3	5	Simplex	70	23
Apr/3	4	3	5	Simplex	72	22.5
Apr/4	4	3	5	Simplex	68	23
May/1	2	3	5	Graph	100	28
May/2	2	3	5	Graph	100	27.5
May/3	2	3	5	Graph	100	29
May/4	3	3	5	Simplex	78	24
May/5	2	3	5	Graph	100	26.5
Jun/1	2	3	5	Graph	98	28.5
Jun/2	3	3	5	Simplex	80	25
Jun/3	2	3	5	Graph	98	29
Jun/4	2	3	5	Graph	100	29.5