Prove Proncelet's Theorem via Resultant

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Abstract

In this paper we will prove Poncelet's Theorem for triangles. To be specific, we consider two conics where one conic is in interior of the other. We prove the existence conditions for a triangle that is circumscribed about interior conic, and also inscribed in the exterior conic. Moreover, we show that if the conditions are satisfied, then there exist infinitely many such triangles. Our approach consists of tow steps: first, we give an explicit condition for a line through two points on the exterior conic to be tangent to interior; then we prove the existence of Poncelet's triangles by using concept of resultant.

Key words: Poncelet's chainlength 3-Poncelet's Triangle, Resultant.

1. Intoduction

Jean-Victor Poncelet (1788-1867) was a French engineer and mathematician who regenerated and made tremendous input into projective geometry. One of his well knowand important works for projective geometry was a Poncelet's Closure Theorem also known as Poncelet's Porism, which states:"Suppose that E_0 is an ellipse in the plane and E_1 is another ellipse that contains E_0 in its interior. If there is one *n*-gon P that is both inscribed in E_1 and circumscribed about E_0, then there is an infinite number of such *n*-gons. (In fact, any point on E_1 is a vertex of exactly one such *n*-gon.)"[1]

Poncelet used synthetic approach of proving Poncelet's Porism. The synthetic style of proofs became predominant in projective geometry in 19th century.Poncelet proved his theorem in 1813, and since that time Poncelet's Theorem was re-approached and proven again by many others. For instance, Jacobi proved Poncelet's Porism in 1828. In modern days, Griffiths and Harris have been proved Poncelet's Theorem in 1977. Their proof was done in algebro-geometrical manner.[3]

Special cases of Poncelet's Porism have been derived many years before the actual prove.For instance, Fuss derived formulas for cases of bicentric quadrilateral, pentagon, hexagon, heptagon, and octagon in 1792.

In this paper we will approach Poncelet's Porism for n=3 from a different prospective. In order to derive the proof, we will use concept of resultant, which is an important tool in Elimination Theory.

2. Background Information

Before we will approach the proof of the Poncelet's triangle, let's look at some definitions and properties of Resultant.

Definition1. [2] Given polynomials $f, g \in k[x]$ of degree l and m of the form:

$$f=a_0x^l+\ldots+a_l, \text{ where } a_0\neq 0, \quad g=b_0x_m+\ldots+b_m, \text{ where } b_0\neq 0.$$

Then the Sylvester matrix of f and g with respect to x is denoted $Syl(f, g, x)$, is the coefficient matrix of the following $(l+m)\times(l+m)$ matrix:

The resultant of f and g with respect to x, denoted $Res(f, g, x)$, is the determinant of the Sylvester matrix. Thus, $Res(f, g, x) = det(Syl(f, g, x))$.

$$\mathrm{Syl}(f,g,x) = \begin{pmatrix} a_0 & 0 & 0 & 0 & b_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & 0 & 0 \\ a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & a_2 & \ddots & a_0 & \vdots & b_2 & \ddots & b_0 \\ a_l & \vdots & \ddots & a_1 & b_m & \vdots & \ddots & b_1 \\ 0 & a_l & \ddots & a_2 & 0 & b_m & \ddots & b_2 \\ 0 & 0 & \ddots & \vdots & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & a_l & 0 & 0 & 0 & b_m \end{pmatrix}$$

Property of Resultant 2. [2]

Given $f, g \in k[x]$ of positive degree, the resultant $Res(f, g, x) \in k$ is an integer polynomial in the coefficients of f and g. Furthermore, f and g have a common factor in $k[x]$ if and only if $Res(f, g, x)=0$.

Property of Resultant 3. [2]

Let $f, g \in k[x_1, \ldots, x_n]$ have positive degree in x_1. Then:
(i) $Res(f, g, x_1)$ is the first elimination ideal $<f, g> \cap k[x_2, \ldots, x_n]$.
(ii) $Res(f, g, x_1)=0$ if and only if f and g havea common factor in $k[x_1, \ldots, x_n]$ which haspositive degree in x_1.

Definition 4.[4]If C, D are different conics, and P_1 is any point on C, one can draw a tangent line on l_0 to from C to D. Let P_1 be the point at which l_0 meets C again. Repeating this, we have for any positive integer m a sequence of points $P_0, \ldots, P_m$in C, $l_i = P_iP_{i+1}$for $0 \leq i \leq m$. This sequence is called Poncelet's chain of length m.
The tangent lines define an algebraic correspondence T on C:

$$T=\{(P, Q) \in C \times C : l = PQ \in D^* = \text{the set of tangents of D}\}.$$

Figure 1 shows a Poncelet's chain of length three, or Poncelet's triangle.

Poncelet's 3 Chain - Poncelet's Triangle

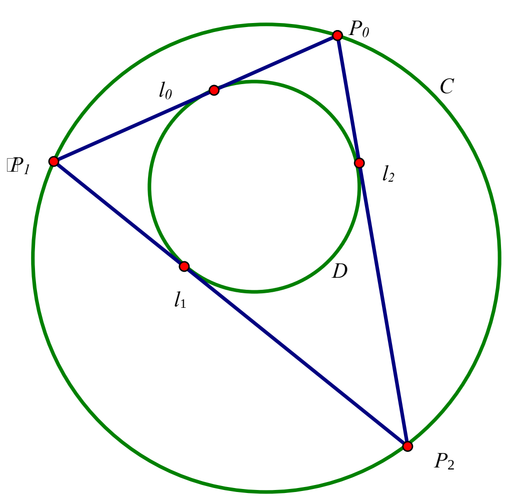

Figure 1

Theorem 5. [4] If two conics C, D in $\mathbb{P}^2$ are defined by
C :$y = x^2$, and D : $c_1x^2 + c_3xy + c_2y^2 + c_4x + c_5y + c_6$=0
Then the algebraic correspondence T on D_0 is defined by $A_1(x, z) = 0$, where $A(x, z) = a_6 + a_4xz + a_1x^2z^2 + a_5(x + z) + a_2xz(x + z) + a_3(x + z)^2$,
where
$a_1 = -4c_1c_2 + c_3^2$, $\quad a_2 = -2(2c_2c_4 + c_3c_5)$,$a_3 = c_5^2 - 4c_2c_6$,
$a_4 = -2c_3c_4 - c_1c_5$, $\quad a_5 = 2(c_4c_5 + 2c_3c_6)$,$a_6 = c_4^2 - 4c_1c_6$.

Proof: Let C and D be two conics.Let P∈ C be a point.Draw a tangent line *l* form P to D, and let Q be the point of intersection *l*∩C. If P∈ C ∩ D, then Q = P, otherwise, Q≠P.Let's draw two tangent lines l_1, l_1' from P, and obtain two points Q_1, Q_1'.Thus P→{Q_1, Q_1'} defines the algebraic correspondence T on C:

$$T=\{(P, Q) \in C \times C : l = PQ \in D^* \}.$$

Now, our goal is to find the defining equation of T.For simplicity, let C :$y = x^2$, and D : $c_1x^2 + c_2y^2 + c_3xy + c_4x + c_5y + c_6$ =0. Then line *l* through points P=(u, u^2) and Q=(v, v^2) on C, tangent to D has slope of$\frac{u^2-v^2}{u-v} = \frac{(u-v)(u+v)}{u-v} = u + v$.

Hence, the equation of *l* is:$y = (u + v)x - uv$.
The intersection of the line *l* and conic D are the solutions to the following equation $c_1x^2 + c_2[(u + v)x - uv2+c3xu+vx-uv+c4x+c5u+vx-uv$
$+c_6$ =0,which can be simplified as

$$(c_1 + c_3u + c_2u^2 + c_3v + 2c_2uv + c_2v^2)x^2 + (c_4 + c_5u + c_5v - c_3uv - 2c_2u^2v - 2c_2uv^2)x + (c_6 - c_5uv + c_2u^2v^2) = 0.$$

Since *l* is tangent to D, then the discriminant of the above quadric equation is
$$D(u, v) = (c_4 + c_5u + c_5v - c_3uv - 2c_2u^2v - 2c_2uv^2)^2 - 4(c_1 + c_3u + c_2u^2 + c_3v + 2c_2uv + c_2v^2)(c_6 - c_5uv + c_2u^2v^2) = 0.$$

Since u, v arbitrary, we replace$u$ and v by x, z respectively, and
$$D(x, z) = (c_4 + c_5x + c_5z - c_3xz - 2c_2x^2z - 2c_2xz^2)^2 - 4(c_1 + c_3x + c_2x^2 + c_3z + 2c_2xz + c_2z^2)(c_6 - c_5xz + c_2x^2z^2)=0$$

ExpandingD(x, z) and simplifying the expansion via Mathematica, we get:

After simplifyingresult in Appendix Iwe get:
$$\begin{aligned} D(x,z) &= (-4\,c_1c_2 + c_3^2)x^2z^2 + (-4\,c_2c_4 + 2\,c_3c_5)(z + x)zx + \left(c_5^2 - 4\,c_2c_6\right) \times (z + x)^2 + (-2\,c_3\,c4 + 4\,c_1c_5)x\,z + (2\,c_4c_5 - 4\,c_3c_6)(x + z) + (c_4^2 - 4\,c_1c_6) \\ &= (-4\,c_1c_2 + c_3^2)x^2z^2 + (-2)(2\,c_2c_4 - c_3c_5)(z + x)\,zx + (c_5^2 - 4\,c_2c_6)(z + x)^2 + (-2c_3c_4 + 4\,c_1c_5)x\,z + (2\,c_4c_5 - 4\,c_3c_6)(x + z) + (c_4^2 - 4\,c_1c_6). \end{aligned}$$

Finally, we rename this polynomial as
$A(x, z) = a_1x^2z^2 + a_2xz(x + z) + a_3(x + z)^2 + a_4xz + a_5(x + z) + a_6$,
where
$a_1 = -4c_1c_2 + c_3^2$, $\quad a_2 = -2(2c_2c_4 + c_3c_5)$, $\quad a_3 = c_5^2 - 4c_2c_6$,
$a_4 = -2c_3c_4 - c_1c_5$, $\quad a_5 = 2(c_4c_5 + 2c_3c_6)$, $\quad a_6 = c_4^2 - 4c_1c_6$.
Thus, we prove the claim.■

3. Prove Poncelet's Theorem Via Resultant

Now we are ready to prove the existence of Poncelet's triangle via resultant.In Figure 2, let point P_0=(x, x^2) on conic C, and construct a tangent to conic D, then this tangent will meet a conic C again at the point P_2=(z, z^2).

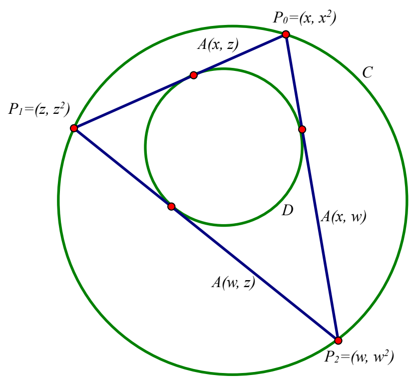

Figure 2

Now let's construct a second tangent to conic D from point P_2=(z, z^2) to point P_3=(w, w^2). Next,let's connect points $P_1$and P_2. We want to show that line $P_2P_0$is also tangent to a conic D. We note that the tangents P_0P_1 and P_1P_2can be described by equations

A(x, z) $=a_1x^2z^2+a_2xz(x+z)+a_3(x+z)^2+a_4xz+a_5(x+z)+a_6$,
A(w, z) $=a_1w^2z^2+a_2wz(w+z)+a_3(w+z)^2+a_4wz+a_5(w+z)+a_6$.
The resultant of A(x, z) and A(z, w) with respect to z, is a polynomial in x , w; of the following form after simplifying:

A(x, w)= $b_1x^2w^2+b_2xw(x+w)+b_3(x+w)^2+b_4xw+b_5(w+z)+b_6$=0,
where coefficients b_1, b_2, b_3, b_4, b_5, and b_6 can be determined as follows:
$b_1=-a_2^2a_3^2+4a_1a_3^3-a_2^2a_3a_4+4a_1a_3^2a_4+a_1a_3a_4^2+a_2^3a_5-4a_1a_2a_3-a_1a_2a_4\ a_5+a_1^2a_5^2$.
$b_2= -a_2a_3^2a_4+2a_1a_3^2a_5+2a_1a_3a_4\ a_5-a_1a_2a_5^2+a_2^2a_6-4a_1a_2a_3\ a_6-a_1a_2a_4a_6+2a_1^2a_5a_6$,
$b_3=a_3^4-a_2a_3^2a_5+a_1a_3a_5^2+a_2^2a_3a_6-2a_1a_3^2a_6-a_1a_2a_5\ a_6+a_1^2a_6^2$,
$b_4= -4a_3^4-4a_3^3a_4-a_3^2a_4^2+6a_2a_3^2a_5+2a_2a_3a_4\ a_5-2a_2^2a_5^2+a_1a_4a_5^2$
$-4a_1a_3^2a_6+a_2^2a_4a_6-4a_1a_3a_4\ a_6-a_1a_4^2a_6+2a_1a_2a_5\ a_6$,
$b_5= -a_3^2a_4a_5+a_1a_5^3+2a_2a_3^2a_6+2a_2a_3a_4\ a_6-a_2^2a_5a_6-4a_1a_3a_5\ a_6-a_1a_4a_5\ a_6+2a_1a_2a_6^2$,
$b_6= -a_3^2a_5^2-a_3a_4a_5^2+a_2a_5^2+4a_3^2a_6+4a_3^2a_4\ a_6+a_3a_4^2a_6-4a_2a_3\ a_6-a_2a_4a_5\ a_6+a_2^2a_6$.

By the geometric meaning of the resultant, i.e., the common factor of the two polymonials,
A(x, w) $=b_1x^2w^2+b_2xw(x+w)+b_3(x+w)^2+b_4xw+b_5(x+w)+b_6$ =0,
isthe condition for the lineP_2P_0 to be tangent to conic D. We also check A(x,w) is in the exact format of A(x,z) with different coefficients. Hence, $\Delta P_0P_1P_2$ is a Poncelet's triangle. ■

4. Conclusion

In summary, we have shown in this paper, first, the conditions for a line through two points on conic C to be tangent to conic D; second, the conditions for the existence of a Poncelet's triangle; finally, if there exists one Poncelet's triangle, than there exist infinitely many of such triangles, since variables x, w, and z are arbitrary.

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