Ring of Invariants Systems with Linear Part $N_{(3)^n}$

Grace Gachigua
Kimathi University College of Technology Department
P.O Box 657-00100, Nyeri, Kenya

David malonza
Kenyatta University
Department of Mathematics
P.O Box 43884-00100, Nairobi, Kenya

Johana Sigey
Jomo Kenyatta University of Agriculture and Technology PAM Department
P.O Box 62000-00100, Nairobi, Kenya

Abstract

The problem of describing the normal form of a system of differential equations at equilibrium with nilpotent linear part is solvable once the ring of invariants associated to the system is known. Our concern in this paper is to describe ring of invariants of differential system with nilpotent linear part made up of n 3×3 *Jordan blocks which is best described by giving the Stanley decomposition of the ring. An algorithm based on the notion of transvectants from classical invariant theory is used to determine the Stanley decomposition for the ring of invariants for the coupled systems when the Stanley decompositions of the Jordan blocks of the linear part are known at each stage.*

Keywords: invariants, box product, transvectant, Stanley decomposition, normal form.

1. *Introduction*

In the study of the qualitative properties of a nonlinear differential equation, $\dot{x} = Ax +$ higher order terms, near an equilibrium point, one of the most powerful techniques available is to simplify A using nonlinear changes of coordinates which leave the origin fixed. The simplified vector field is called the normal form of A. The theory of normal form is concerned with finding the simplest form for the system by removing as many terms as possible with the remaining ones having dynamical significance.

There are well-known procedures for putting a system of differential equations

$$\dot{x} = Ax + v(x) \qquad (1.1)$$

(where v is a formal power series with quadratic terms) into normal form with respect to its linear part, A. The normal form theory divides into two parts, the case when A=S is diagonalizable and the case when A=N is nilpotent, that is nilpotent systems. The general case can be solved by combining the results of the two special cases. The goal of this paper is to describe the ring of invariants of a differential system (1.1) when its linear part A is a nilpotent matrix N, where

$$N = \begin{bmatrix} N_3 & & \\ & \ddots & \\ & & N_3 \end{bmatrix} \quad \text{and} \quad N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Our main result is a procedure that solves the description problem where N, the nilpotent matrix is in Jordan form, with coupled n Jordan blocks, provided that the description problem is already solved for each Jordan block of N taken separately. Our method is based on adding one block at a time.

The problem of finding Stanley decomposition for the equivariants of $N_{22\ldots2}$ was first solved by Cushman et al.,[3] using a method called "covariants of special equivariants". Their method begins by creating a scalar problem that is larger than the vector problem and their procedures derived from classical invariant theory thus it was necessary to repeat calculations of classical theory at the levels of equivariants. Malonza [4] solved the same problem by "Groebner" basis methods found in Adams et al., [1] rather than borrowing from classical results. Murdock and Sanders [7], developed an algorithm based on the notion of transvectants to determine the normal form of a vector field with a nilpotent linear part, when the normal form is known for each Jordan block of the linear taken separately. The algorithm is based on the notion of transvectant, from classical invariant theory. Malonza, [5] using the algorithm in [6] for transvectants computed the Stanley decomposition for Takens-Bogdanov systems and his results agreed with his previous work in [4].

Namachchivaya et al [8], studied a generalized Hopf bifurcation with non-semisimple 1:1 Resonance. The normal form for such a system contains only terms that belong to both the semisimple part of linear part and the normal form of the nilpotent, which is a couple Takens-Bogdanov system with linear part,

$$A=\begin{bmatrix} i\omega & 1 & & \\ & i\omega & & \\ & & i\omega & 1 \\ & & & i\omega \end{bmatrix}.$$

This example illustrates the physical significance of the study of normal forms for systems with nilpotent linear part.

Our results are mainly based on the work found in [6], that is, the application of tranvectant's method (also known as box product) for computing Stanley decompositions for the ring of invariants of nilpotent systems.

2. Invariants and Stanley decomposition.

Let $\mathscr{P}_j(R^n,R^m)$ denote the vector space of homogeneous polynomials of degree j on R^n with coefficients in R^m, where R denotes the set of real numbers. Let $\mathscr{P}(R^n,R^m)$ be the vector space of all such polynomials of any degree and let $\mathscr{P}_*(R^n,R^m)$ be the vector space of formal power series. If m=1, $\mathscr{P}_*(R^n,R^m)$ becomes the ring of formal power series on R^n, where R denotes the set of real numbers. For such smooth vectors fields, it is sufficient to work with polynomials.

For the nilpotent matrix N, define the lie operator

$$L_N:\mathscr{P}_j(R^n,R^n)\to\mathscr{P}_j(R^n,R^n) \text{ by}$$

$$(L_N v)=v'(x)Nx-Nv(x) \qquad (2.1)$$

and the differential operator

$$D_{Nx}:\mathscr{P}_j(R^n,R)\to\mathscr{P}_j(R^n,R)\text{, by}$$

$$(D_{Nx}f)(x)= f'(x)N(x)= (N(x).\nabla)f(x). \qquad (2.2)$$

Then D_N is a derivation of the ring $\mathscr{P}(R^n,R)$, meaning that

$$D_N(fg)=(D_N f)g+f(D_N g). \qquad (2.3)$$

In addition,

$$L_N(fv) = (D_N f)v + fL_N v \qquad (2.4)$$

A function is called an invariant if $\frac{¶}{¶t} f(e^{Nt}(x))\big|_{t=0} = 0$ or equivalently $D_N f = 0$ or $f\,Î\,\ker D_N$.

Since

$$D_N(f+g) = D_N f + D_N g$$
$$D_N(fg) = fD_N g + gD_N f\,,$$

It follows that if f and g are invariants, then so are f + g and fg; that is ker D_N is both a vector space over R and also a subring of $\mathscr{P} = (R^n, R)$ know as the *ring of invariants*.

Similarly a vector field *v* is called an *equivariants* of *Ax*, if $\frac{\partial}{\partial t}(e^{-At}v(e^{At}x))\big|_{t=0} = 0$ that is

$v\,Î\,\ker L_A$.

There are two normal form styles in common use for nilpotent systems, the *inner product normal form* and the *sl(2) normal form*. The inner product normal form is defined by $\mathscr{P}(R^n, R^n) = imL_N \oplus \ker L_{N^*}$, where N^* is the conjugate transpose of N, a nilpotent matrix. To define the *sl(2)* normal form, one first set $X = N$ and construct matrices Y and Z such that

X, Y and *Z* satisfy the relation

$$[X,Y] = Z, [Z,X] = 2X, [Z,Y] = -2Y. \qquad (2.5)$$

An example of such an sl(2) triad is

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Having obtained the triad {*X, Y, Z*} we create two additional (induced) triads {$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$} and {$\mathsf{X}, \mathsf{Y}, \mathsf{Z}$} as follows

$$\mathcal{X} = D_Y \qquad \mathcal{Y} = D_X \qquad \mathcal{Z} = D_Z \qquad (2.6)$$
$$\mathsf{X} = \mathsf{L}_Y \qquad \mathsf{Y} = \mathsf{L}_X \qquad \mathsf{Z} = \mathsf{L}_Z \qquad (2.7)$$

where the first is a triad of differential operators and the second is a triad of lie operators. The name *sl(2)* normal form style results from the fact that {*X, Y, Z*} span a lie algebra of $n \times n$ matrices isomorphic to the Lie algebra *sl(2*). Observe that the operators {X,Y,Z} map each $\mathscr{P} = (R^n, R)$ into itself. It then follows from the representation theory of *sl(2)*in [5] that

$$\mathscr{P} = (R^n, R) = \mathrm{imY} \oplus \mathrm{kerX} = \mathrm{imX} \oplus \mathrm{kerY}$$

Clearly the $\ker \mathcal{X}$ is a subring of $\mathscr{P} = (R^n, R)$, the ring of invariants and it follows from that ker X is a module over this subring. This is the *sl(2)* normal form module.

The most effective way of describing the invariant ring associated with a nilpotent matrix *N* is by a device from commutative algebra called a *Stanley decomposition*, introduced for this purpose in [7]. We write $R[[x_1, ..., x_n]]$ for the ring of (scalar) formal power series in $x_1, ..., x_n$. A *subalgebra* $\Re$ of $R[[x_1, ..., x_n]]$ is a subset that is both a subring and a vector subspace. The subalgebra is *graded* if

$$\Re = \bigoplus_{d=0}^{\infty} \Re_d\,,$$

where $\Re_d$ is the vector subspace of $\Re$ consisting of elements of degree d. To define a *Stanley decomposition* of a graded subalgebra, we begin with the definition of a Stanley term.

A *Stanley term* is an expression of the form $R[[f_1,...,f_n]]\varphi$ where the elements $f_1,...,f_n$ and φ are homogeneous polynomials and $f_1,...,f_n$ not including φ) are required to be algebraically independent. The Stanley term $R[[f_1,...,f_n]]\varphi$ denotes the set of all expressions of the form $F(f_1,...,f_n)\varphi$, where *F* is a formal power series in k variables. When $\varphi=1$, φ is omitted, and the Stanley term is a subalgebra, otherwise it is only a subspace. A *Stanley decomposition* is a finite direct sum of Stanley terms. The algebraic independence and direct sum conditions in the definition of a Stanley decomposition imply that each element of the subalgebra has a unique expression in the form dictated by the Stanley decomposition. A *doubly graded Stanley decomposition is* graded by degree and weight thus a polynomial f is called *doubly homogeneous of type* (d,w) if every monomial in f has degree d and weight w. Weights are integers, and unlike degrees can be negative, but invariants cannot have negative weights. A vector subspace *V* of $\ker\mathscr{X}$ is *doubly graded* if

$$V=\bigoplus_{d=0}^{\infty}\bigoplus_{w=0}^{\infty}V_{dw},$$

where V_{dw}, is the vector subspace of V consisting of doubly homogeneous polynomials of degree d and weight w.

A *standard monomial* associated with a Stanley decomposition is an expression of the form $f_1^{m1},...,f_k^{mk}\varphi$, where $R[[f_1,...,f_n]]\varphi$ is a term in the Stanley decomposition. Notice that "monomial" here means a monomial in the basic invariants, which are polynomials in the original variables $x_1,...,x_n$. Given a Stanley decomposition of $\ker\mathscr{X}$, its standard monomials of a given degree (or of a given type) form a basis for the (finite-dimensional)vector space of invariants of that degree (or type). Next, we give Stanley decompositions for rings of invariants associated with N_2 and N_3 using the notion in [4]. The ring of invariants of N_2 in $R[x_1,y_1]$ is $\ker\mathscr{X}_2$. This ring clearly contains

which is of type (1,1), and in fact every element of $\ker\mathscr{X}_2$ can be written uniquely as a formal power series $f(x_1)$ in x_1 alone. We express this by the Stanley decomposition

$$\ker\mathscr{X}_2=R[[\alpha]]$$

The ring of N_{22} in $R[[x_1,x_2,y_1,y_2]]$ is described by the Stanley decomposition

$$\ker\mathscr{X}_{22}=R[[\alpha_1,\alpha_2,\beta_{12}]]$$

with $\alpha_1=x_1,\alpha_2=x_2,\beta_{12}=x_1y_2-x_2y_1$.. Here α_1 and α_2 are of type (1,2) and β is of type (2,0).

To prove that we have obtained all the invariants (transvectants), we need to generate the table function of the Stanley decomposition. We replace each term of the decomposition by a rational function P/Q in d and w (d for degree and w for weight) construct as follows: for each basic invariants appearing inside the square brackets, the denominator will contain a factor $1-d^pw^q$, where p and q are the degree and weight of the invariants; the numerator will be P/Q, where p and q are the degree and weight of the standard monomials of that term. When the rational function P/Q from each term of the Stanley decomposition are summed up we obtain the table function T given by $T=\sum_i P_i/Q_i$. . Thus, for examples above, the table function are $T_2=\dfrac{1}{1-dw}$ and $T_{22}=\dfrac{1}{(1-dw)(1-d^2)}$.

The following lemma found in [6] gives a method to check that enough basic invariants have been found.

Lemma 1: Let {X,Y,Z} be a triad of $n\times n$ matrices, let {$\mathscr{X},\mathscr{Y},\mathscr{Z}$} be the induced triad and suppose that $I_1,...,I_n$ is a finite set of polynomials in ker $\mathscr{X}$.

Let R be a subring of $R[I_1,...,I_n]$; suppose that the Stanley terms have been found, and that the Stanley decomposition and its associated table function $T(d,w)$ have been determined. Then $R = \ker \mathcal{X} \subset \mathcal{P}(R^n, R^n)$ if and only if $\left.\frac{\partial}{\partial w}(wT)\right|_{w=1} \frac{1}{(1-d)^n}$.

3. Box Products of Stanley Decompositions

Let V_k, k = 1, 2, be *sl(2)* representation spaces with triads $\{X_k, Y_k, Z_k\}$. Then $V_1 \boxtimes V_2$ is a representation space with triad $\{X, Y, Z\}$, where $X = X_1 \boxtimes I + I \boxtimes X_2$ (and similarly for *Y* and *Z*). We now define the *box product* by

$$(\ker X_1 \boxtimes \ker X_2) = \ker X \ . \tag{3.1}$$

To begin to put the box product into computationally useful form, we use the notion of transvetants introduced for this purpose in [6]. Consider a system with nilpotent linear part

$$N = \begin{pmatrix} \hat{N} & 0 \\ 0 & \tilde{N} \end{pmatrix}$$

where $\hat{N}$ and $\tilde{N}$ are nilpotent matrices of sizes $\hat{n} \times \hat{n}$ and $\tilde{n} \times \tilde{n}$ respectively ($\hat{n} + \tilde{n} = n$, in (upper) Jordan form, and each may consist of one or more Jordan blocks). Let $\{\mathcal{X},\mathcal{Y},\mathcal{Z}\}$, $\{\hat{\mathcal{X}},\hat{\mathcal{Y}},\hat{\mathcal{Z}}\}$ and $\{\tilde{\mathcal{X}},\tilde{\mathcal{Y}},\tilde{\mathcal{Z}}\}$ be the associated triads of operators. Notice that the first triad acts on $R[[x_1,...,x_n]]$, the second on $R[[x_1,...,x_{\hat{n}}]]$ and the third on $R[[x_{\hat{n}+1},...,x_n]]$.

Suppose that $f = f(x_1,...,x_n) \in \ker \hat{\mathcal{X}}$ and $g = g(x_{\hat{n}+1},...,x_n) \in \ker \tilde{\mathcal{X}}$ are weight invariants of weights w_f and w_g, and *i* is an integer in the range $0 \le i \le \min(w_f, w_g)$. Then we define *external transvectant* of f and *g* of order *i* to be the polynomial $(f,g)^i \in R[[x_1,...,x_n]]$ given by

$$(f,g)^i = \sum_{j=0}^{i} (-1)^j W_{f,g}^{i,j} (\hat{\mathcal{Y}}^j f)(\hat{\mathcal{Y}}^{j-1} g), \tag{3.2}$$

where

$$W_{f,g}^{i,j} = \binom{i}{j} \frac{(w_f - j)!}{(w_f - i)!} . \frac{(w_g - i + j)!}{(w_g - i)!}$$

We say that a transvectant $(f,g)^i$ is *well-defined* if *i* is in the proper range for *f* and *g*. Notice that the zeroth transvectant is always well-defined and reduces to the product: $(f,g)^0 = fg$. Given Stanley decompositions for ker $\ker \hat{\mathcal{X}}$ and ker $\tilde{\mathcal{X}}$, the following theorem provides a basis for $\ker \mathcal{X}$ in each degree which is a first step toward obtaining a Stanley decomposition for $\ker \mathcal{X}$.

Theorem 2: *Each well-defined transvectant* $(f,g)^i$ *of f* ∈ $\ker \hat{\mathcal{X}}$ *and g* ∈ ker $\tilde{\mathcal{X}}$ *belongs to* ker $\mathcal{X}$. If *f and g are doubly homogeneous polynomials of types* (*df, wf*) *and* (*dg, wg*) *respectively,* $(f,g)^i$ *is a doubly homogeneous polynomial of type* (*df* + *dg, wf* + *wg* − 2*i*). *Suppose that Stanley decompositions for* $\ker \hat{\mathcal{X}}$ *and are* ker $\tilde{\mathcal{X}}$ *given. Then a basis for the (finite-dimensional) subspace* (ker $\mathcal{X}$)*d of homogeneous polynomial in* $\ker \mathcal{X}$ *with degree d is given by the set of all well-defined transvectants* $(f,g)^i$ *where f is a standard monomial of the Stanley decomposition for* $\ker \hat{\mathcal{X}}$, *g is a standard monomial of the Stanley decomposition for* ker $\tilde{\mathcal{X}}$ *and df* + *dg* = *d.*

The proof of this theorem is given in section 6 of [4] and will not be repeated here. The bases given by Theorem 2 are sufficient to determine ker $\mathcal{X}$ one degree at a time, but to find all of ker $\mathcal{X}$ in this way would require finding infinitely many transvectants. A Stanley decomposition for ker $\mathcal{X}$ must be based on a finite number of basic invariants. To construct such a decomposition, we must first find an alternative basis for each (ker X) *d* that uses only a finite number of transvectants overall. (We do not count zeroth transvectants, which are simply products. A Stanley decomposition can produce an infinite number of products). Such alternative bases can be found by the following replacement theorem found in [6].

Theorem 3. *Any transvectant* $(f,g)^i$ *in the basis given by Theorem 1 can be replaced by a product* $(f_1,g_1)^{i_1}\dots(f_j,g_j)^{i_j}$ *of transvectants, provided that* $f_1\dots f_j=f$, $g_1\dots g_j=g$ *and* $i_1+\dots+i_j=i$.
The following corollary of the Replacement Theorem will play a crucial role in our calculations.

Corollary 4: *If* $w_h=w_k=r$ *so that* $(h,k)^{(r)}$ *has weight zero, then whenever* $(fh,gk)^{(i+r)}$ *is well-defined, it may be replaced by* $(f,g)^{(i)}(h,k)^{(r)}$.

Proof. Clearly $(fh,gk)^{(i+r)}$ and $(f,g)^{(i)}(h,k)^{(r)}$ have the same stripped form and total transvectant order. It is only necessary to observe that $(f,g)^{(i)}$ is well-defined. But $w_{fh}=w_f+w_h=w_f+r\ge i+r,$ so $w_f\ge i$ and similarly $w_g\ge i.$

The next lemma is now trivial, but essential to the method.
Lemma 5: *Box distributes over direct sums of admissible subspaces: If* $\hat{V}\subset\ker\hat{\mathcal{X}}, \tilde{V}_1\subset\ker\tilde{\mathcal{X}}$ *,and* $\tilde{V}_2\subset\ker\tilde{\mathcal{X}}$ *are admissible subspaces, with* $\tilde{V}_1\cap\tilde{V}_2$, *then* $\tilde{V}_1\oplus\tilde{V}_2$ *is admissible and* $\hat{V}\boxtimes(\tilde{V}_1\oplus\tilde{V}_2)=(\hat{V}\boxtimes\tilde{V}_1)\oplus(\hat{V}\boxtimes\tilde{V}_2),$ *and similarly for* $(\tilde{V}_1\oplus\tilde{V}_2)\boxtimes\hat{V}.$
We complete this section by the following theorem which is Theorem (9) in [6], and outlines the procedure for computing $\ker\mathcal{X}$

Theorem 6. *A Stanley decomposition of* $\ker\mathcal{X}=\ker\hat{\mathcal{X}}\boxtimes\ker\tilde{\mathcal{X}}$ *is computable in a finite number of steps given Stanley decomposition of* $\ker\hat{\mathcal{X}}$ *and* $\ker\tilde{\mathcal{X}}$.
Proof. The proof is given in [6] but we briefly outline ideas in the proof important in our calculations. By Lemma 5, we can compute $\ker\mathcal{X}$ if we can compute any box product of the form $R[[f_1,\dots,f_k]]\varphi\boxtimes R[[g_1,\dots,g_l]]\psi$, where each factor is a Stanley term from the given decompositions of $\ker\hat{\mathcal{X}}$ and $\ker\tilde{\mathcal{X}}$.

Let p be the number of elements of weight > 0 in $f_1,\dots,f_k$ and q the number of such elements in $g_1,\dots,g_l$. We proceed by double induction on p and q.

Suppose $p = q = 0$. Then the box product is spanned by transvectants of the form $(f_1^{m1},\dots,f_1^{mk}\varphi,g_1^{n1},\dots,g_l^{nl}\psi),$, which is well-defined if and only if $0\le i\le r$, where $r=\min(w_\varphi,w_\psi)$. (The *f* and *g* factors add no weight, and cannot support any higher transvectants.) By Theorem 3 each transvectant may be replaced by $f_1^{m1}\dots f_k^{mk},g_1^{n1}\dots g_l^{nl}(\varphi,\psi)^i$, which remains well-defined. Therefore

$$R[[f_1,\dots,f_k]]\varphi\boxtimes R[[g_1,\dots,g_l]]\psi\cong\oplus_{i=0}^{r}R[[f_1,\dots,f_k,g_1,\dots,g_l]](\varphi,\psi)^i.$$

Now we make the induction hypothesis that all cases with $p = 0$ are computable up through the case q− 1, and we discuss case *q*. Choose one of the *q* elements of $g_1,\dots,g_l$ having positive weight; we assume the chosen element is g_1. Then we may expand

$$R[[g_1,...,g_l]]\psi = (\bigoplus_{v=0}^{t-1} R[g_2,...,g_l]]\psi) \oplus R[[g_1,...,g_l]]g_1^t\psi.$$

where t is the smallest integer such that $w_{g_1^t\varphi} > w_\psi$. This decomposition corresponds to classifying monomials according to the power of g_1 that occurs, with all powers greater than or equal to t assigned to the last term. Now take the box product of $R[[f_1^{m1},...,f_k^{mk}]]\varphi$ times this expression, and distribute the product according to Lemma 5. All of the terms except the last are computable by the induction hypothesis we claim the last term is computable by the formula

$$R[[f_1,...,f_k]]\varphi \boxtimes R[[g_1,...,g_l]]g_1^t\psi \cong \bigoplus_{i=0}^{w_\varphi} R[[f_1,...,f_k,g_1,...,g_l]](\varphi, g_1^t\psi)^i.$$

This is because w_φ is an absolute limit to the order of transvectants in this box product that will be well-defined, and any such transvectant $(f_1^{m1}...f_k^{mk}\varphi, g_1^{n1}...g_l^{nl}g_1^t\psi)^i$ can be replaced by

$$f_1^{m1}...f_k^{mk}\varphi, g_1^{n1}...g_l^{nl}(g_1^t\varphi,\psi)^i$$

Now we make the induction hypothesis that cases ($p-1$, q), (p, $q-1$), and ($p-1$, $q-1$) can be handled, and we treat the case (p, q). Choose one of the p functions in $f1, . . . , fk$ having positive weight; we assume the chosen element is f_1. Similarly, choose a function of positive weight from $g_1,...,g_l$ and suppose it is g_1. Let s and t be the smallest integers such that $s.w_{f_1} = t.w_{g_1}$

Expand

$$R[[f_1,...,f_k]]\varphi = (\bigoplus_{u=0}^{s-1} R[f_2,...,f_k]]\varphi) \oplus R[[f_1,...,f_k]]f_1^s\varphi$$

and

$$R[[g_1,...,g_l]]\psi = (\bigoplus_{v=0}^{t-1} R[[g_2,...,g_l]]g_1^v\psi) \oplus R[[g_1,...,g_l]]g_1^t\psi.$$

Taking the box product of these last two expansions and distribute the product. There are four kinds of terms. Terms that are missing both $f1$ and $g1$ in square brackets are of type ($p-1$, $q-1$). Terms that are missing $f1$ in square brackets, but not $g1$, are of type ($p-1$, q), and there are likewise terms of type (p, $q-1$). All of these can be handled by the induction hypothesis. Finally, there is the term

$$R[[f_1,...,f_k]]f_1^s\varphi \boxtimes R[[g_1,...,g_l]]g_1^t\psi$$

There is no upper limit to the transvectant order that can occur here, since in general there remain terms of positive weight in the square brackets. However, setting $r = s.w_{f_1} = t.w_{g_1}$ it can be shown that this box product is equivalent to

$$R[[f_1,...,f_k]]f_1^s\varphi \otimes R[[g_1,...,g_l]]g_1^t\psi \cong$$

$$\bigoplus_{i=0}^{r-1} R[[f_1,...,f_k,g_1,...,g_l]](f_1^s\varphi, g_1^t\psi)^i \oplus (R[[f_1,...,f_k]]\varphi \boxtimes R[[g_1,...,g_l]]\psi)(f_1^s, g_1^t)^r$$

The final term is quite different from any others considered until now, since it involves a box product of subspaces as the coefficient of $(f_1^s, g_1^t)^r$. At this point we have reduced the calculation of $R[[f_1,...,f_k]]\varphi \boxtimes R[[g_1,...,g_l]]\psi$ in case (p,q) to a number of terms computable by the induction hypothesis or by explicit formula, plus one special term that seems to lead in circles since it involves the very same box product that we are trying to calculate. Thus our result has the form

$$\Re = \delta \oplus \Re\theta,$$

where θ has weight zero. But this implies $\Re = \delta \oplus (\delta \oplus \Re\theta)\theta = \delta \oplus \delta\theta \oplus \Re\theta^2$. Continuing in this way we have $\Re = \delta \oplus \delta\theta^2 \oplus \delta\theta^3 \oplus ...$, which reduces to $\Re = \delta[[\theta]]$.

This simply means that we erase the "unusual" term $(R[[f_1,...,f_k]]\varphi \boxtimes R[[g_1,...,g_l]]\psi)(f_1^s,g_1^t)^r$ from our computation, and instead insert $\theta=(f_1^s,g_1^t)^r$ into the square brackets in all the coefficient rings that have already been computed. This does not affect the induction, because the new elements added have weight zero, and the induction is on the numbers p and q of elements of positive weight.

4. *The Ring of Invariants for Coupled* $N_{33...3}$ *Systems*

The ring of invariants of N_3 in R[x,y,z] is ker N_3. Let $N=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$.

The sl(2) triad will be as follows

$$X=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\qquad Y=\begin{bmatrix}0&0&0\\2&0&0\\0&2&0\end{bmatrix}\qquad Z=\begin{bmatrix}2&0&0\\0&0&0\\0&0&-2\end{bmatrix}$$

Having obtained the triad {X,Y,Z}, we create additional triad $\{\mathcal{X},\mathcal{Y},\mathcal{Z}\}$

$$\mathcal{X}=D_Y=2x\tfrac{\partial}{\partial y}+2y\tfrac{\partial}{\partial z}$$
$$\mathcal{Y}=D_X=y\tfrac{\partial}{\partial x}+z\tfrac{\partial}{\partial y}$$
$$\mathcal{Z}=D_Z=2x\tfrac{\partial}{\partial x}-2z\tfrac{\partial}{\partial z}$$

The differential operators $\{\mathcal{X},\mathcal{Y},\mathcal{Z}\}$ map each vector space of homogeneous scalar polynomials into itself, with $\mathcal{X}$ and $\mathcal{Y}$ being nilpotent and $\mathcal{Z}$ semisimple. The eigenvectors of $\mathcal{Z}$ (called weight vectors) are the monomials x^m and the associated eigenvalues(called weights) are $<m,\mu>$ where $\mu=(\mu_1,...,\mu_n)$ are the eigenvalues of $\mathcal{Z}$ that is $\mathcal{Z}(x^n)=<m,\mu>x^m$.

The basic invariants can be shown to be $\alpha=x,\beta=y^2-2xz$. Here α is of degree 1 weight 2 and β is of degree 2 weight 0. Every element of ker $\mathcal{X}_3$ can be written uniquely as a formal series $f[\alpha,\beta]$ in x,y,z. We describe this by the Stanley decomposition ker $\mathcal{X}_3=R[[\alpha,\beta]]$.

4.1 Linear Part N_{33}

From above the Stanley decomposition of $\ker\mathcal{X}_3=R[[\alpha_1,\beta_1]]$ we have that by theorem 6 $\ker\mathcal{X}_{33}=\ker\mathcal{X}_3\boxtimes\ker\tilde{\mathcal{X}}_3$ where $\ker\tilde{\mathcal{X}}_3=R[[\alpha_2,\beta_2]]$ corresponds to the second bracket in N_{33}. Expanding, we have

$$\ker\mathcal{X}_3=R[[\beta_1]]\oplus R[[\alpha_1,\beta_1]]\alpha_1$$
$$\ker\tilde{\mathcal{X}}_3=R[[\beta_2]]\oplus R[[\alpha_2,\beta_2]]\alpha_2.$$

Note that β_1 and β_2 are terms of weight zero and we do not expand along terms of weight zero so they are suppressed and will have them appear in every square bracket of the box product we compute. Therefore

$$\ker\mathcal{X}_{33}=[R\oplus R[[\alpha_1]]\alpha_1]\boxtimes[R\oplus R[[\alpha_2]]\alpha_2]$$

Distributing the box product by considering all well defined transvectants $(f,g)^i$ gives two kinds of terms.

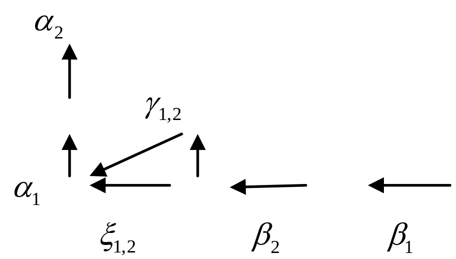
1. Two terms that are immediately computed in final form: $R \oplus R[[\alpha_1]]\alpha_1]$
2. One box product: $R \boxtimes R[[\alpha_2]]\alpha_2 = R[[\alpha]]\alpha.$
3. One box product $R[[\alpha_1]]\alpha_1 \boxtimes R[[\alpha_2]]\alpha_2$. This will recycle to $R[[\alpha_1]] \boxtimes R[[\alpha_2]]$. Indeed:

$$R[[\alpha_1]]\alpha_1 \boxtimes R[[\alpha_2]]\alpha_2 = R[[\alpha_1,\alpha_2]]\alpha_1\alpha_2 \oplus R[[\alpha_1,\alpha_2]](\alpha_1,\alpha_2)^{(1)} \oplus R[[\alpha_1,\alpha_2]](\alpha_1,\alpha_2)^{(2)}.$$

Let $(\alpha_1,\alpha_2)^{(1)} = \gamma_{1,2}$ and $(\alpha_1,\alpha_2)^{(2)} = \xi_{1,2}$. According to recycling rule the last term will be deleted and $\xi_{1,2}$ which has weight zero will be inserted to all square brackets along side the suppressed invariants. Collecting and recombining all the terms, whenever possible we have:

$$\ker \mathcal{X}_{33} = R[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]] \oplus R[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\gamma_{1,2}.$$

The same Stanley decomposition can also be obtained from the lattice diagram below where the Stanley terms are viewed as a sum of the path from β_1 to α_2 with a corner at $\gamma_{1,2}$.



With the monotone paths:

$(\beta_1 \to \beta_2 \to \xi_{1,2} \to \alpha_1 \to \alpha_2)$

$(\beta_1 \to \beta_2 \to \xi_{1,2} \to \alpha_1 \to \alpha_2)\gamma_{1,2}.$

4.2 Linear Part N_{333}

When $n = 3$ the Stanley decomposition for the ring of invariants is given by

$$\ker \mathcal{X}_{333} = \ker \mathcal{X}_{33} \boxtimes \ker \mathcal{X}_3$$
$$=[R[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]] \oplus R[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\gamma_{12}] \boxtimes R[[\alpha_3,\beta_3]]$$

There are two cases to consider. Distributing the box products and recombining terms where Possible we have:

$$\begin{aligned}
\ker \mathcal{X}_{333} = {} & R[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]] \oplus \\
& \mathrm{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{12} \oplus \\
& \mathrm{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{13} \oplus \\
& \mathrm{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{12},\gamma_{13} \oplus \\
& \mathrm{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\gamma_{23} \oplus \\
& \mathrm{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\xi_{2,3} \oplus \\
& \mathrm{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\gamma_{12},\gamma_{23} \oplus \\
& \mathrm{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\gamma_{12},\xi_{2,3} \oplus \\
& \mathrm{R}[[\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]](\alpha_3,\gamma_{12})^{(1)} \oplus \\
& \mathrm{R}[[\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]](\alpha_3,\gamma_{12})^{(2)}.
\end{aligned}$$

To verify that this is the true Stanley Decomposition consider the table function which in this case is given by

$$T_9=\frac{dw^2}{(1-dw^2)^3(1-d^2)^5}+\frac{d^3w^4}{(1-dw^2)^3(1-d^2)^5}+\frac{d^2w^2}{(1-dw^2)^3(1-d^2)^5}$$
$$+\frac{d^4w^4}{(1-dw^2)^3(1-d^2)^5}+\frac{1}{(1-dw^2)^2(1-d^2)^6}+2\frac{d^2w^2}{(1-dw^2)^2(1-d^2)^6}$$
$$+\frac{d^4w^4}{(1-dw^2)^2(1-d^2)^6}+\frac{d^3w^2}{(1-dw^2)(1-d^2)^6}+\frac{d^3}{(1-dw^2)(1-d^2)^6}$$

Multiplying the table function by w, differentiating with respect to w and putting $w=1$ and it can be easily be shown that

$$\frac{\partial}{\partial w}wT_9\Big|_{w=1}=\frac{1}{(1-d)^9}\ .$$

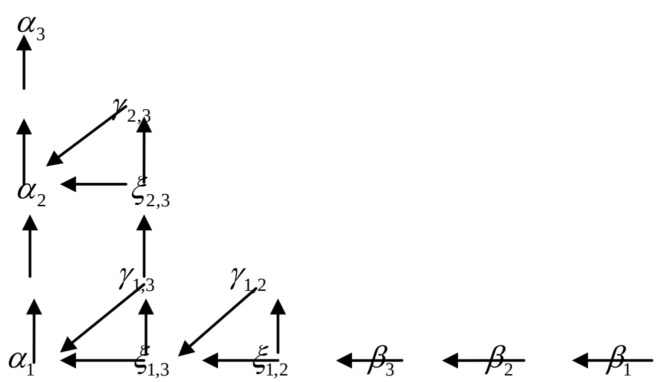
Hence the table function is perfect thus all the tranvectants have been found.
The following observations are made from the Stanley decomposition above:

a) the first term of Stanley decomposition has no product outside the square bracket.

b) the transvectants $\gamma_{1,2},\gamma_{1,3}$ and $\gamma_{2,3}$ never appears inside the square brackets.

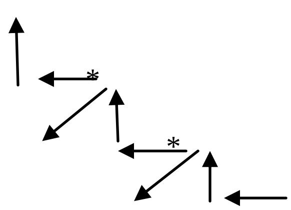
c) the transvectants $\xi_{1,2},\xi_{1,3}$ and $\xi_{2,3}$ appears inside as well as outside the square brackets.

It is evident that the same Stanley decomposition of N_{333} can be obtained from sum of the paths in the following lattice diagram:



Where

- every path takes the form:



We refer to * as a corner.

- each square brackets of the Stanley decomposition contains all invariants in a path except $\gamma_{k,l}$ and the product of transvectants outside the square bracket is the product of the invariants at the corners.
- Stanley decomposition of the ring of invariants $\ker \mathcal{X}_{333}$ is then given by the sum of the terms T_1 and T_2, where
- $T_1=\oplus_j R$ [[invariants on the j^{th} path]](product of corners on the j^{th} path) exiting at α_k and ending at α_3 where k=1,2.
- $T_2=\oplus_j R$ [[invariants on the j^{th} path]](product of corners on the j^{th} path, $\alpha_3)^{(i)}$ exiting at α_2 through $\gamma_{2,3}$ and ending at α_3 and i=1,2.

- From the above examples we conclude that:
- for every additional n, there are new transvectants $(\alpha_k, \beta_l)^{(i)}$ where $i = 1,2$ and $1 \le k < l \le n.$
- the lattice diagram of $N_{(3)^n}$ is obtained by adding these new transvectants together with α_n to the lattice diagram for $N_{(3)^{n-1}}$.
- the first term of the Stanley decomposition has no products of transvectants outside the square brackets.
- the transvectants $(\alpha_k, \beta_l)^{(1)} = \gamma_{1,2}$ where never appears inside the square brackets.
- the transvectants $(\alpha_k, \beta_l)^{(2)} = \xi_{1,2}$ where $1 \le k < l \le n.$ appears inside as well as outside the square brackets.
- the Stanley decomposition of $N_{(3)^n}$ is the sum of terms of T_1 and T_2, where

$T_1 = \oplus_j R$ [[invariants on the j^{th} path]](product of corners on the j^{th} path) exiting at α_k and ending at α_n where $k = 1,...,n-1.$

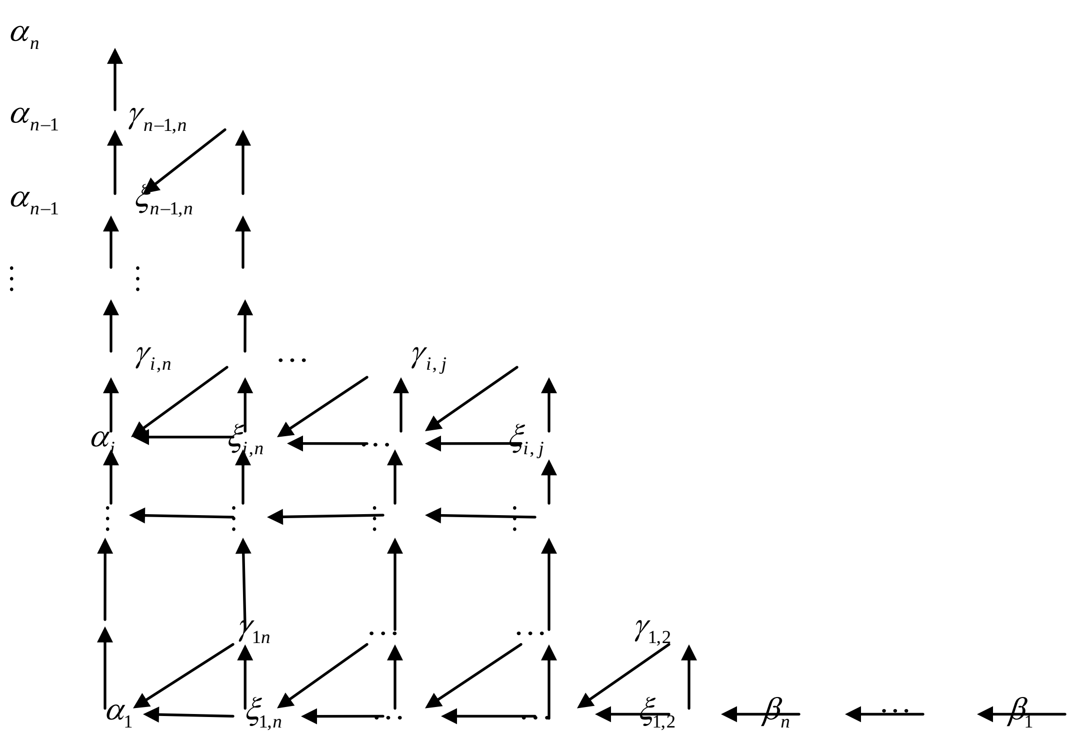
$T_2 = \oplus_j R$ [[invariants on the j^{th} path]](product of corners on the j^{th} path, $\alpha_m)^{(i)}$ exiting at α_m through $\gamma_{m-1,m}$ and ending at α_n where $i = 1,2$ and $m = 3,4,..,n.$

In general, we have the following theorem for obtaining the Stanley decomposition of systems with linear part $N_{(3)^n}$ as:

Theorem 7: The Stanley decomposition of the ring of invariant of $N_{(3)^n}$ is given by the sum of terms T_1 and T_2 where j will range over all possible number of paths for ker $\mathfrak{X}_{(3)^n}$, where

$T_1 = \oplus_j R$ [[invariants on the j^{th} path]](product of corners on the j^{th} path) exiting at α_k and ending at α_n where $k = 1,...,n-1.$

$T_2 = \oplus_j R$ [[invariants on the j^{th} path]](product of corners on the j^{th} path, $\alpha_m)^{(i)}$ exiting at α_m through $\gamma_{m-1,m}$ and ending at α_n where $i = 1,2$ and $m = 3,4,..,n.$



Proof: We prove by induction on $n.$ It is true for $n=2$ and $n=3$, by the above examples. We suppose that it is true for $k=n-1$ and show that it hold for $k=n.$ Since

$$\ker \mathcal{X}_{(3)^n} = \ker \mathcal{X}_{(3)^{n-1}} \boxtimes \ker \mathcal{X}_3$$

Suppressing all transvectants of the form $\beta_1,..,\beta_n$ and $\xi_{k,j}$ for $1 \le k < l \le n.$ since they are of weight zero and noting that they will be added to every square brackets depending on the terms they are found we have: $R[[\alpha_i,..,\alpha_{n-1}]]\varphi \boxtimes R[[\alpha_n]].$

Expanding the box product:

$$R[[\alpha_i,..,\alpha_{n-1}]]\varphi \boxtimes R[[\alpha_n]] = (R[[\alpha_{i+1},...,\alpha_{n-1}]] \oplus R[[\alpha_i,\alpha_{i+1},..,\alpha_{n-1}]]\alpha_i\varphi) \boxtimes (R \oplus R[[\alpha_n]]\alpha_n)$$

Distributing the box product gives three kinds of terms.

1. Two terms that are computed to final form: $R[[\alpha_{i+1},...,\alpha_{n-1}]] \oplus R[[\alpha_i,\alpha_{i+1},..,\alpha_{n-1}]]\alpha_i\varphi$
2. One box product: $R[[\alpha_{i+1},..,\alpha_{n-1}]]\varphi \boxtimes R[[\alpha_n]]\alpha_n,$ that must be computed by further expansions.
3. One box product that recycles: $R[[\alpha_i,\alpha_{i+1},..,\alpha_{n-1}]]\alpha_i\varphi \boxtimes R[[\alpha_n]]\alpha_n =$

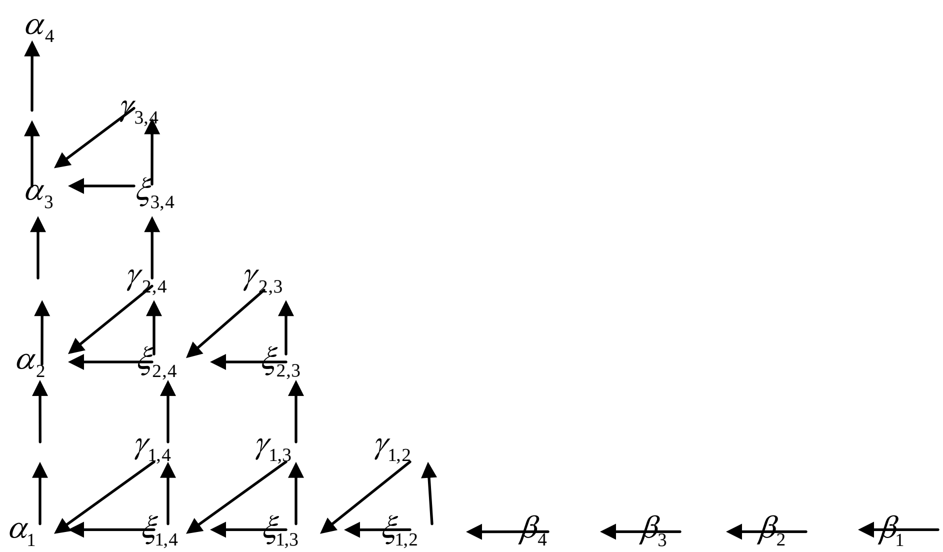
$$R[[\alpha_i,\alpha_{i+1},...,\alpha_{n-1},\alpha_n]]\alpha_i\varphi \oplus R[[\alpha_i,\alpha_{i+1},..,\alpha_{n-1},\alpha_n]](\alpha_i\varphi)^{(1)} \oplus [R[[\alpha_i,\alpha_{i+1},..,\alpha_{n-1}]]\varphi \boxtimes R[[\alpha_n]]](\alpha_i,\alpha_n)^{(2)}.$$

We delete the last term and insert $(\alpha_i,\alpha_n)^{(2)} = \xi_{i,n}$ in square brackets together with other suppressed tranvectants. Recombining terms whenever possible, we finally find the Stanley decomposition of $\ker \mathcal{X}_{(3)^n}$.

Equivalently, finding all the additional transvectants for $\ker \mathcal{X}_{(3)^n}$ of the form $\gamma_{i,n}$ and $\xi_{i,n}$ where $1 \le i < n$ and adding these together with α_n to the lattice diagram of $\ker \mathcal{X}_{(3)^{n-1}}$ we obtain the lattice diagram for $\ker \mathcal{X}_{(3)^n}$ as shown above and the sum of the j^{th} paths of the form T_1 and T_2 gives the Stanley decomposition of $\ker \mathcal{X}_{(3)^n}$ as required. □

We summarize our work by applying Theorem 7 in finding the Stanley decomposition for the ring of invariants with linear part N_{3333}. .

The new transvectants created are $\gamma_{1.,4}\gamma_{1,3},\gamma_{1,4},\xi_{1,4},\xi_{2,4},\xi_{3,4}.$ By adding these transvectants and α_4 to the lattice diagram for N_{333} , we illustrate how to get the Stanley decomposition from the lattice diagram below:



The Stanley decomposition of the ring of invariants $\ker \mathcal{X}_{3333}$ is then given by the sum of the terms of $\oplus_j T_1$ and $\oplus_j T_2$. Let $\Re = R[[\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3}]],$ the final results are:

$$
\begin{aligned}
\ker \mathcal{X}_{3333} &= \Re[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]] \oplus \Re[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,2} \oplus \\
&\Re[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,3} \oplus \Re[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,4} \oplus \\
&\Re[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,2}\gamma_{1,3} \oplus \Re[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,2}\gamma_{1,4} \oplus \\
&\Re[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,3}\gamma_{1,4} \oplus \Re[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{1,4} \oplus \\
&\Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{2,4} \oplus \Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{2,4} \oplus \\
&\Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,3}\gamma_{2,4} \oplus \Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{2,4} \oplus \\
&\Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\xi_{2,4} \oplus \Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\xi_{2,4} \oplus \\
&\Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,3}\xi_{2,4} \oplus \Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{1,3}\xi_{2,4} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{3,4} \oplus \Re[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{3,4} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,3}\gamma_{3,4} \oplus \Re[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{3,4} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\xi_{3,4} \oplus \Re[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\xi_{3,4} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,3}\xi_{3,4} \oplus \Re[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{1,3}\xi_{3,4} \oplus \\
&\Re[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2},\alpha_4)^{(1)} \oplus \Re[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi(\gamma_{1,2},\alpha_4)^{(2)} \oplus \\
&\Re[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,3},\alpha_4)^{(1)} \oplus \Re[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi(\gamma_{1,3},\alpha_4)^{(2)} \oplus \\
&\Re[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2}\gamma_{1,3},\alpha_4)^{(1)} \oplus \Re[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi(\gamma_{1,2}\gamma_{1,3},\alpha_4)^{(2)} \oplus \\
&\Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{2,3} \oplus \Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{2,3}\gamma_{2,4} \oplus \\
&\Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{1,2}\gamma_{2,3} \oplus \Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{1,2}\gamma_{2,3}\gamma_{2,4} \oplus \\
&\Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\xi_{2,3} \oplus \Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{2,4}\xi_{2,3} \oplus \\
&\Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{1,2}\xi_{2,3} \oplus \Re[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{1,2}\xi_{2,3}\gamma_{2,4} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]]\gamma_{2,3}\gamma_{3,4} \oplus \Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{2,3}\gamma_{3,4} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]]\gamma_{2,3}\xi_{3,4} \oplus \Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{2,3}\xi_{3,4} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]]\xi_{2,3}\gamma_{3,4} \oplus \Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\xi_{2,3}\gamma_{3,4} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]]\xi_{2,3}\xi_{3,4} \oplus \Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\xi_{2,3}\xi_{3,4} \oplus \\
&\Re[[\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]](\gamma_{2,3},\alpha_4)^{(1)} \oplus \Re[[\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]](\gamma_{2,3},\alpha_4)^{(2)} \oplus \\
&\Re[[\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2}\gamma_{2,3},\alpha_4)^{(1)} \oplus \Re[[\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2}\gamma_{2,3},\alpha_4)^{(2)} \oplus \\
&\Re[[\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2}\xi_{2,3},\alpha_4)^{(1)} \oplus \Re[[\alpha_4,\xi_{2,3},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2}\xi_{2,3},\alpha_4)^{(2)} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{3,4}]](\gamma_{1,2},\alpha_3)^{(1)} \oplus \Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{3,4}]](\gamma_{1,2},\alpha_3)^{(2)} \oplus \\
&\Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{3,4}]](\gamma_{1,2}\gamma_{3,4},\alpha_3)^{(1)} \oplus \Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{3,4}]](\gamma_{1,2}\gamma_{3,4},\alpha_3)^{(2)} \oplus \\
&\Re[[\alpha_4,\xi_{2,3},\xi_{3,4}]](\gamma_{1,2}\gamma_{3,4},\alpha_3)^{(1)} \oplus \Re[[\alpha_3,\alpha_4,\xi_{2,3},\xi_{3,4}]](\gamma_{1,2}\gamma_{3,4},\alpha_3)^{(2)} \oplus \\
&\Re[[\alpha_4,\xi_{2,3},\xi_{3,4}]]((\gamma_{1,2},\alpha_3)^{(1)},\alpha_4)^{(1)} \oplus \Re[[\alpha_4,\xi_{2,3},\xi_{3,4}]]((\gamma_{1,2},\alpha_3)^{(1)},\alpha_4)^{(2)}.
\end{aligned}
$$

To verify that this a Stanley decomposition consider the table function which in this case is given by

$$T_{12}=\frac{w+3d^2w^3+3d^4w^5+d^6w^7}{(1-dw^2)^4(1-d^2)^7}+\frac{2d^2w^3+4d^4w^5+2d^6w^7+2d^2w+4d^4w^3+2d^6w^5}{(1-dw^2)^3(1-d^2)^8}+$$
$$\frac{d^2w^3+3d^4w^5+2d^6w^7+d^2w+4d^4w^3+3d^6w^5+d^4w+d^6w^3}{(1-dw^2)^2(1-d^2)^9}+$$
$$\frac{3d^3w^3+3d^3w+2d^5w^5+3d^5w^3+d^5w}{(1-dw^2)(1-d^2)^9}+\frac{d^3w^3+d^3w+d^5w^5+d^5w^3}{(1-dw^2)^2(1-d^2)^8}+$$
$$\frac{d^4w^3+d^4w}{(1-dw^2)(1-d^2)^8}.$$

Multiplying the table function by w, differentiating with respect to w and putting $w=1$ and it can be easily be shown that

$$\frac{\partial}{\partial w}wT_{12}\big|_{w=1}=\frac{1}{(1-d)^{12}}..$$

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