# Selected General Characteristics of Proportional Guidance Trajectories

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## *Introduction*

Proportional guidance method is currently the most common method of guiding of guided missiles, especially those that are intended to destroy quick maneuvering targets. The reason is not only relatively small curving of proportional convergence trajectories, but mostly fairly well mastered technology self-guided warhead implementation for measuring of basic flight control parameter: absolute angular velocity of aimed target rotation.

In available literature, the issue of proportional guidance is discussed particularly in [1], [2], [3], [4], where, findings from a number of other undisclosed parameters are probably summarized. The publication, however, still miss a complete and accurate assessment of some general characteristics of proportional guidance kinematics that can be drawn from relatively simple analysis of system of kinematic equations, considering ideal bond equations without the need for their numerical integration. This area is also targeted by article content.

In analyzing the kinematics of plane proportional guidance of object N to the target object C, we accept for designation of kinematic parameters symbolism according to Fig.1.

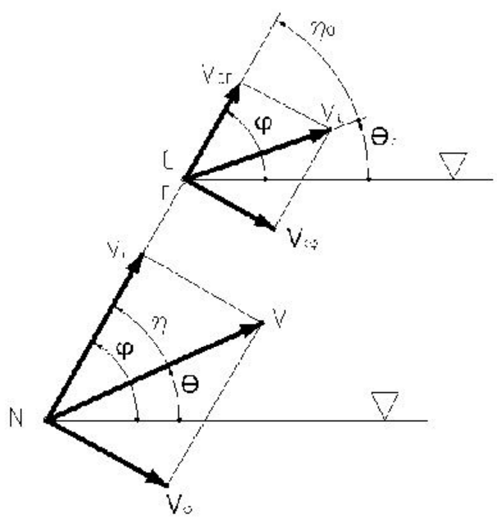


**Fig. 1 the symbolism of the kinematics of plane proportional guidance of object N to the target object C**

Ideal bond equation of proportional guidance method then can be written in form:

$$\dot{\theta} = k\dot{\varphi} \qquad (1)$$

Where coefficient k is so called guidance constant

For simplicity, we accept an agreement that the direction of velocity vector of target object VC identifies with the reference axis, i.e. we put $\theta_c = 0$ and thus $\eta_c = \varphi$.
According to Fig. 1 we simply check that the kinematic parameters of target and guided object are linked together, by kinematic equations:

$$\dot{r} = V_c cos\varphi - V cos\eta = V_{cr} - V_r \qquad (2)$$

$$r\dot{\varphi} = V sin\eta - V_c sin\varphi = V_\varphi - V_{c\varphi} \qquad (3)$$

## 1. *Existence, number, type and kinematic stability of rectilinear trajectories of proportional guidance*

We are considering an ideal bond equation (1), which implies for absolute rectilinear trajectory of proportional guidance from the obvious requirement $\theta$ = const. and at the same time also the condition

$$\dot{\varphi} = 0$$

Thus from other kinematic equation (3) further equality of vertical components of velocity vectors of guided and target object is

$$V_\varphi = V_{c\varphi} \qquad (4)$$

Therefore it is evident, that rectilinear trajectory of proportional guidance always coincides with a trajectory parallel guidance and can only exist at constant speed ratio of both objects

$$q = \frac{V}{V_c} = cons$$

And for certain aiming-off allowance $\eta^*$, satisfying at position angle $\varphi = \varphi^*$ condition

$$sin\eta^* = \frac{1}{q} sin\varphi^* \qquad (5)$$

Note that we will continue to refer to kinematic parameters that belong to rectilinear trajectory by right superscript *.
The number and type of rectilinear trajectories is closely related to the size of the speed ratio q:

a) For q = const. > 1 satisfy the condition (5) for every $\varphi^* \in < 0, \pi >$ exist exactly two values of aiming-off allowance $\eta^* \in < 0, \pi >$

$$\eta_a^* = arc\ sin\left(\frac{1}{q} sin\varphi^*\right) < \frac{\pi}{2} \qquad (6)$$

$$\eta_b^* = \pi - \eta_a^* > \frac{\pi}{2}$$

Therefore, in this case, for each $\varphi^* \in < 0, \pi >$ there are exactly two different absolute rectilinear trajectories of proportional guiding, shown in Fig. 2 by straight lines $a_1$, $b_1$, ($a_2$, $b_2$) for situation of arrival (departure) of target object.

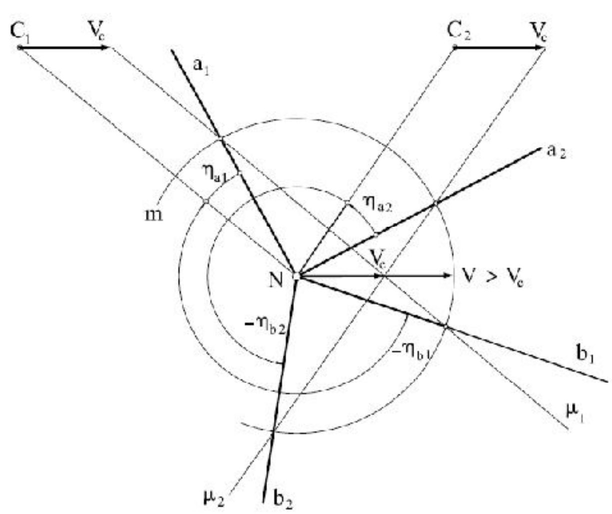


**Fig. 2 Two different absolute straightforward trajectories of proportional guidance**

Construction of straight lines $a_1$, $b_1$ position is based on the consideration that only in intersection of the circle m with straight line ui, i = 1, 2 is equality (4) satisfied. From graphic representation is simultaneously obvious that for q > 1 are trajectories $a_i$ ($b_i$) always convergence (recede) trajectories.

b) For q = const. < 1 it is possible to satisfy condition (5) only for angles of sight

$$\varphi^* \leq Are \sin q$$

When for each of them there will be again just two rectilinear trajectories with size of aiming-off allowance given by relations (6).

From Fig. 3, is obvious that for

- $\varphi^*$ ϵ < 0, arc sin q > there will be two different trajectories of proportional recede,
- $\varphi^*$ ϵ < π, arc sin q, π > on the other hand, there will be two different trajectories of proportional convergence,
- $\varphi^*$ ϵ < 0, arc sin q each of stated pairs of trajectories merge in a single one with aiming-off allowance $\eta_a = \eta_b = \frac{\pi}{2}$

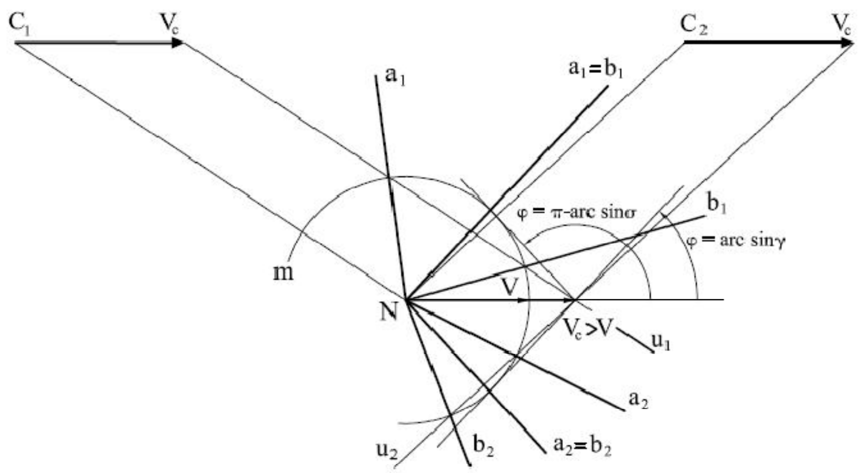


**Fig. 3 Two rectilinear trajectories with the size of the run**

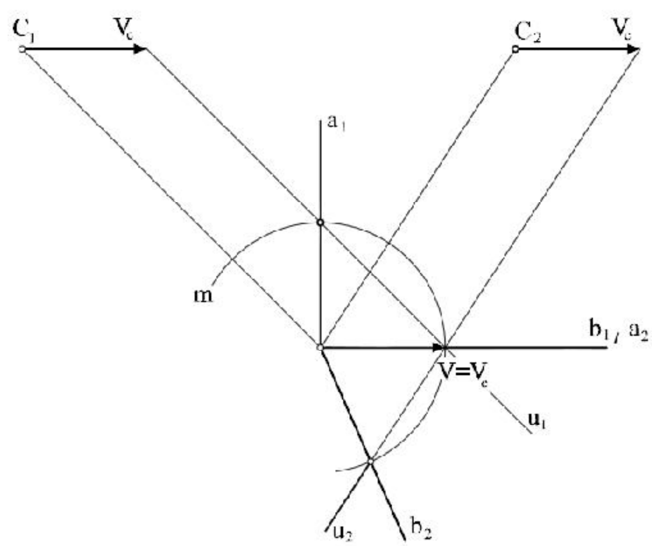


**Fig. 4 Rectilinear trajectories when b1 (a2) line is parallel to the rectilinear trajectory of the target object**

c) If q = const. = 1 for every $\varphi^* \epsilon < 0, \pi >$ exist again exactly two rectilinear trajectories, whereby one of which, in Fig. 4 straight line $b_1$ ($a_2$) will be parallel to rectilinear trajectory of target object (while maintaining constant distance of both objects), and the second for

$-\varphi^* \epsilon \left(\frac{\pi}{2}, \pi >\right.$ Will be trajectory of convergence (straight line $a_1$),

$-\varphi^* \epsilon < 0, \frac{\pi}{2}$ Will be trajectory of recede (straight line $b_2$).

For $\varphi^* = \frac{\pi}{2}$ both rectilinear trajectories merge in one, parallel to trajectory of target object.

From the above analysis, inter alia, is obvious that the number and type of rectilinear trajectories of proportional guidance in no way depend on the size of guiding constant. Before we proceed to evaluation of kinematic stability of rectilinear trajectories of proportional guidance, note that it is understood as such characteristic of trajectory, when the object guided to it is going back along deviation.

The emergence of deviation from rectilinear trajectory can be generally understood as a condition where the initial parameter values $\varphi_0$, $\eta_0$ do not satisfy condition (5). Some general characteristics of curved trajectory, which corresponds to movement along deviation can be determined from analysis of the second kinematic equation (3), considering integral of ideal bond equation (1).

By integrating (1) we get:

$$\theta - \theta_0 = k(\varphi - \varphi_0) \qquad (7)$$

From Fig.1 follows $\theta = \varphi - \eta$ and also $\theta_0 = \varphi_0 - \eta_0$ and dependence of aiming-off allowance $\eta$ size and components of speeds $V_\varphi$, $V_{c\varphi}$ on size of angle $\varphi$ and initial conditions $\varphi_0$, $\eta_0$, it is therefore possible for proportional guidance based on the relation (7) to be expressed in the form:

$$\eta = (1-k)\varphi + \eta_0 - (1-k)\varphi_0 \qquad (8)$$

$$V_\varphi = V\sin\eta = V\sin[(1-k)\varphi + \eta_0 - (1-k)\varphi_0] \qquad (9)$$

$$V_{c\varphi} = V_c \sin\varphi \qquad (10)$$

Dependency of components $V_\varphi$, $V_{c\varphi}$ size on angle $\varphi$ and size of aiming-off allowance $\eta$ is for $k > 2$, $q > 1$ shown in Fig. 5 considering, that according to relation (8), size of aiming-off allowance $\eta$ for $k > 1$ with rising $\varphi$ declines.

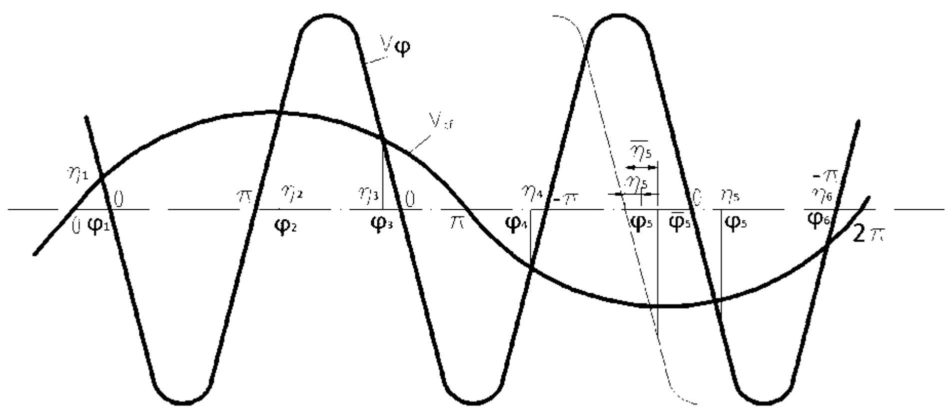


**Fig. 5 Dependence of $V_\varphi$, $V_{c\varphi}$ components on angle and size of aiming-off allowance η**

From equation (8) can be easily derived that to intersection of graph $V_\varphi(\varphi)$ with axis $\varphi$, when aiming-off allowance $\eta$ has value 0, $\pm\eta$, belongs size of angle of sight

$$\varphi_\eta = 0 = \varphi_0 + \frac{\eta_0}{k-1} \qquad (11)$$

$$\varphi_\eta = \pm\pi = \varphi_\eta - 0 \pm \frac{\pi}{k-1} \qquad (12)$$

and from physical aiming-off allowance obviously continues to apply:

$$\eta_0 \equiv \eta_0 \pm i.2\pi, \qquad i = 0, 1, 2, \ldots$$

The difference ($V_\varphi$ - $V_{c\varphi}$) is according to kinematic equation (3) equal to value $r_\varphi$. Mutual intersections of graphs $V_\varphi$ (φ) and $V_{c\varphi}$ (φ) in Fig. 5 therefore correspond for practically interesting case r ≠ 0 to such parameter values $\varphi_i$, $\eta_i$, i = 1, 2, ..., when φ = 0 and movement trajectory is rectilinear. These intersections form at the same time limits of value φ intervals alteration, in which is $V_\varphi > V_{c\varphi}$ and also φ > 0 with value φ intervals, in which is $V_\varphi < V_{c\varphi}$ and also φ < 0.

In process of proportional guidance thus, while keeping the ideal bond, parameter φ, η value is always indefinitely close to some - depending on the initial values $\varphi_0$, $\eta_0$ – from these values $\varphi_0^*, \eta_0^*$, which creates left border of interval, in which is φ < 0 (i.e. simultaneously right border of interval, in which is φ > 0). In Fig. 5 these values have even indexes i. For rectilinear trajectories with parameters $\varphi_j^*, \eta_j^*$- in Fig. 5 is index j odd – which creates right border of interval $\dot{\varphi} < 0$ (i.e. simultaneously left border of interval, in which is $\dot{\varphi} > 0$) apparently no curvilinear trajectory cannot convert.

It can be easily check, that for considered case, when q > 1, k > 2 it is about proportional recede trajectory.

When evaluating kinematic stability or instability of rectilinear trajectory of proportional guidance, based on the above considerations, it is impossible to forget the important fact that parameters φ, η are converting from initial values $\varphi_0$, $\eta_0$ to values $\varphi_i^*, \eta_i^*$ only if equality (9) is not satisfied for them – in Fig. 5 relation (12), (11) – i.e. if applies

$$\Delta\eta_0 = (1-k)\Delta\varphi_0 \qquad (13)$$

where $\Delta\eta_0 = \eta_0 - \eta_0^*$

$\Delta\varphi_0 = \varphi_0 - \varphi_0^*$

In like manner, if guided object deviates from rectilinear trajectory with parameters $\varphi_i^*, \eta_i^*$ so, that deviations $\Delta\varphi_0$, $\Delta\eta_0$ don't satisfy relation (13), it doesn't come back to its initial rectilinear trajectory, but its curvilinear trajectory converts to different rectilinear trajectory with parameters $\overline{\varphi_i^*}$, $\overline{\eta_i^*}$, which satisfies relation (13). For example in deviation from rectilinear trajectory with parameters $\varphi_5^*, \eta_5^*$, Fig. 5 to state with parameters $\varphi_5, \eta_5$ - curvilinear trajectory will be after deviation converted to different rectilinear trajectory, which have parameters $\overline{\varphi_5^*}$, $\overline{\eta_5^*}$.

Kinematic stability or instability of given rectilinear trajectory of proportional guidance must thus always be assessed with considering whether for deviations Δφ0, Δη0, equality (13) is or isn't satisfied. Given analysis simultaneously indicates incorrectness and inaccuracy of certain statements in [1] by which the number of rectilinear trajectories depends on the size of guiding constant and these trajectories divide space around target object to certain sectors in which are only curvilinear trajectories.

## 2. *Influence of size of guiding constant k and speeds ratio q on character of curvilinear trajectories*

Kinematic equation (3) can be written in form:

$$(r^* + \Delta r)(\dot{\varphi}^* + \Delta\dot{\varphi}) = V\,sin(\eta^* + \Delta\eta) - V_c sin(\dot{\varphi}^* + \Delta\varphi) \qquad (3)$$

Where: $\Delta\varphi = \varphi(t) - \varphi^*$

$\Delta\eta = \eta(t) - \eta^*$

$\Delta r = r(t) - r^*(t)$

Are deviations of parameters $\varphi(t)$, $\eta(t)$, $r(t)$ curvilinear trajectories from parameters $\varphi^*$, $\eta^*$, $r^*(t)$ of reference rectilinear trajectory? If in equation (3) trigonometric functions are distributed in Taylor series and if we neglect assuming we have a small degree of deviation of members with their second and higher powers and also their mutual conjunction after adjustment we obtain

$$(r^* + \Delta r).\dot{\varphi}^* + r^*.\Delta\dot{\varphi} = (V\,sin\eta^* - V_c sin\varphi^*) + (V\,cos\eta^*.\Delta\eta - V_c cos\varphi^*.\Delta\varphi)$$

If we consider, that for rectilinear trajectory applies

$$\dot{\varphi}^* = V sin\eta^* - V_v sin\varphi^* = 0$$

Mutual bond of small deviations $\Delta\varphi$, $\Delta\eta$ can be expressed from parameters $\varphi^*, \eta^*$ of reference rectilinear trajectory by linear diferential equation

$$r^*.\Delta\varphi = \dot{V}\,cos\eta^*.\Delta\,\eta - V_c cos\varphi^*.\Delta\varphi \qquad (14)$$

We designate size of variables at the beginning of guidance, when t = 0, as

$$r = r_0 = r_{0;}^* \;\; \Delta\varphi_0 = \varphi_0 - \varphi^*; \;\; \Delta\eta = \Delta\eta_0 - \eta_0 - \eta^*$$

For proportional guidance, changes in parameters φ, η are bounded by relations (7), (8) and therefore for their increments (deviations) at t> 0 apparently applies

$$(\Delta\varphi - \Delta\varphi_0)(1-k) = \Delta\eta - \Delta\eta_0.$$

Using this relation, we adjust the equation (14) to a form suitable for its quadrature

$$r^*.\Delta\dot{\varphi} + a\left(\Delta\varphi - \frac{b}{a}\right) = 0 \qquad (14)$$

where

$$a = V cos\eta^*(1-k) - V_c cos\varphi^* = \dot{r^*} + V.k.cos\eta^*,$$
$$b = V cos\eta^*[\Delta\eta_0 - \Delta\varphi_0(1-k)].$$

If we consider that for a rectilinear trajectory applies

$$\dot{r^*} = V_c cos\varphi^* - V\ cos\eta^* = const$$

and therefore also

$$r^* = r_0 + \dot{r^*}.t,$$

General solution of differential equation (14) can be written in form

$$\Delta\varphi = \left(\Delta\varphi_0 - \frac{b}{a}\right)\left(\frac{r^*}{r_0}\right)^{\frac{k}{\%}-1} + \frac{b}{a} \qquad (15)$$

where

$$\chi = \frac{\dot{r^*}}{V cos\eta^*} \qquad (16)$$

After adjustment, we obtain relation for size of angle φ

$$\varphi = \overline{\varphi^*} + \overline{\Delta\varphi_0}\left(\frac{r^*}{r_0}\right)^{\frac{k}{\chi}-1} \qquad (17)$$

Where

$$\overline{\varphi^*} = \varphi^* + \frac{b}{a},$$
$$\Delta\overline{\varphi_0} = \varphi_0 - \overline{\varphi^*}.$$

From derived relations (15), (17), (14) is clear, that in deviation from rectilinear trajectory of convergence, where $\dot{r^*} < 0$ (and also for $r^* \to 0$), if condition

$$k > \chi \qquad (18)$$

is satisfied,

- deviation Δφ will convert to value $\frac{b}{a}$,
- inclination angle of relative trajectory φ will convert to value $\overline{\varphi^*}$,
- Difference of speed components $V_\varphi - V_{c\varphi} = r^*.\dot{\varphi} = r^*.\Delta\dot{\varphi}$ will convert to zero.

At the time of encounter, when r = 0, parameters φ, η acquire such values $\overline{\varphi^*}, \overline{\eta^*} = \eta_0 + \Delta\overline{\varphi_0}(1-k)$, which belong to a certain rectilinear trajectory. If initial deviations $\Delta\varphi_0$, $\Delta\eta_0$ satisfy equality (13), apparently applies: $\overline{\varphi^*} = \varphi^*$, $\overline{\eta^*} = \eta^*$, i.e. curvilinear trajectory converts to the initial reference rectilinear trajectory. Summing up the results of the analysis so far, including knowledge from point 1, it is possible to express the sentence: When V, $V_c$, $\theta_c$ = const., k > 1 each kinematic curvilinear trajectory of proportional convergence converts to certain rectilinear trajectory, with which merges at the very moment of encounter.

The condition (18) can be considered as a criterion for evaluation of kinematic stability of initial - reference rectilinear trajectory with parameters $\varphi^*$, $\eta^*$ in terms of whether the rectilinear trajectory is coming back to it or not, in case of deviations which satisfy equality (13). When condition (18) is not satisfied, size of deviations with flight time (for $\dot{r^*} < 0$) is rising. Relations, derived from linearized kinematic equation therefore affect in this case tendency of changes only at the beginning of guiding and never in its end, i.e. at the stage of encounter. Kinematic curvilinear trajectory, however, also in this case converts to certain rectilinear trajectory.

Note that to achieve kinematic stability of rectilinear trajectory of proportional recede (i.e. for $\dot{r}^* > 0$, size of guiding constant vice-versa as shown in (15), (17) must be limited downward

k < $\chi$

By adjusting (16) we get relation

$$\chi = 1 + \frac{cos\varphi^*}{\sqrt{q^2 - sin^2\varphi^*}} \qquad (19)$$

in which sign „-„ is valid for $|\eta *| < \frac{\pi}{2}$ and sign „+" for $|\eta *| > \frac{\pi}{2}$.

Dependence of factor % size on ratio of speed q and inclination angle of relative rectilinear trajectory $\varphi^*$ is shown in Fig. 6.

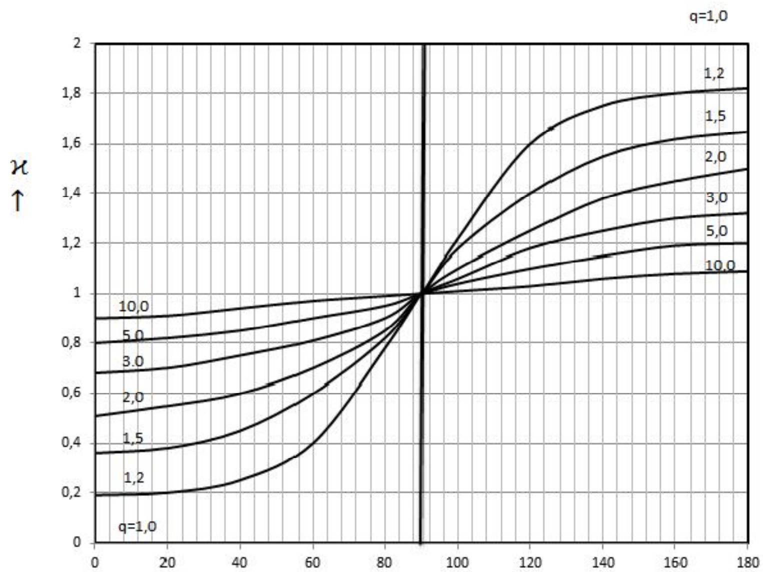


**Fig. 6 Dependence of the factor % at the speed ratio q and relative rectilinear trajectory angle φ ***

For flight on a curved trajectory, guided object performs certain lateral maneuver Δθ, which size is possible - within the limits of linearization accuracy – to assess, considering (1) by relation

$$\Delta\theta = k.\Delta\overline{\varphi_0} \qquad (20)$$

If desirable (correct) size of aiming-off allowance $\eta^*$ was at the beginning of guidance when $\varphi_0 = \varphi^*$ adhered with abberation $\Delta\eta_0$, after adjustment from (20) follows

$$\frac{\Delta\theta}{\Delta\eta_0} = \frac{\frac{k}{\%}}{\frac{k}{\%} - 1} \qquad (21)$$

Diagram in Fig. 7 documents the beneficial effect of increasing guiding constant k to reduction of total size of direction maneuver, particularly at lower k values and during accommodate guiding, when$|\varphi_0| > 90°$. At the same time, contradictory effect of increasing speed ratio q is evident.

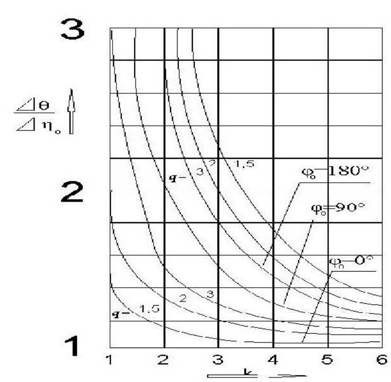


**Fig. 7 The effect of increasing the drawing-constant on reduction of the total size of directional maneuver**

Curvature of trajectory and demands on maneuverability of guided object are directly proportional to size of angular velocity $\dot{\theta}$. Influence of guiding constant k size to the curvature of proportional guidance trajectory can be therefore assessed by relation:

$$\dot{\theta} = k\dot{\varphi} = k\dot{\varphi}_0 \left(\frac{r^*}{r_0}\right)^{\frac{k}{\%}-2} \qquad (22)$$

Where

$$\dot{\varphi}_0 = \frac{V_c.\sqrt{q^2 - sin^2\dot{\varphi}_0}}{r_0}\Delta\eta_0$$

Resulting from derivation of (17).

If size of guiding constant satisfies condition

$$k > 2\chi \qquad (23)$$

Maximum curvature of proportional convergence kinematic trajectory is at its beginning and gradually decreases to zero at the encounter.

Character of kinematic trajectory curvature change is shown in Fig. 8. The diagram shows that with growth of guiding constant k, unevenness of curvature distribution also grows: maximum of curvature is increasing even though that overall size of direction maneuver Δθ, as shown in Fig. 7 decreases.

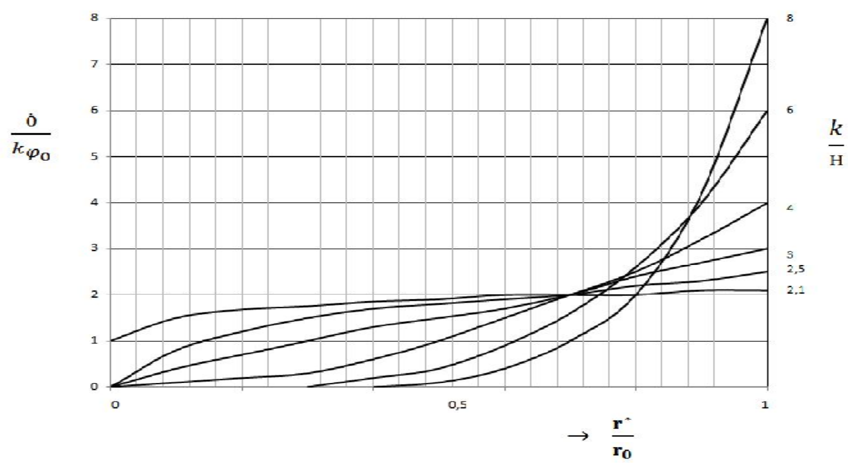

**Fig. 8 Change of kinematic trajectory curvature**

At $\chi < 2\chi$ otherwise curvilinear kinematic trajectory of convergence as well converts to reference rectilinear trajectory, but its curvature constantly rises and at encounter rises above all limits. In this connection, let us note that real size of angular velocity $\dot{\theta}$, which guided object can create is always limited and therefore in this case at given moment occurs permanent decrease in real size of guiding constant and thus also disappearance of convergency to rectilinear trajectory of convergence and creation of minutium.

## *Conclusion*

The results of numerical solution of proportional guiding trajectories on computer are in agreement with stated general characteristics of curvilinear trajectories.

## *Literature*

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