Complex Numbers As Visual Representation

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Abstract

There are many useful connections between complex numbers and visual representations. It has been presented a theorem deal with diagonals of parallelogram using complex numbers in the study.

Key Words: Complex numbers, visual representations

1.1.Complex Numbers

Complex numbers, as a basic component of the functions of complex variable theory, have an effect on all areas of mathematics. In addition to the elegant structure and importance in mathematics, complex numbers are one of the most common mathematical concepts used by theoretical and applied mathematics. Although with a certain importance of aesthetic, it has very strong connections among the sub-branches of mathematics For example, the fundamental theorem of algebra"A polynomial equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x+a_n=0$, where $a_k\in C(k = 0,1,2,\ldots,n)$, $a_0\neq 0$, $n\geq 1$, has a solution in C" has been proven in a very elegant way using the properties of complex numbers.

Complex numbers is one of the topics taught/learned in analytical and prescriptive approach traditionally. In recent years, despite numerous researches indicating the importance of graphics and visualization, approaches for complex numbers have remained fairly analytical. According to Cuoco (1997), the development of the complex number system C in precalculus or algebra textbooks usually follows one of the two paths:

- ***i.*** the existence of an "imaginary unit i"
- ***ii.*** the complex numbers are "defined" by giving R^2 an algebraic structure in which addition is defined as the ordinary vector addition and multiplication is defined by the formula (a,b).(c,d)=(ac-bd, ad+bc).

In addition, in the third way, complex numbers that are introduced in the pairs of real number (a,0) is isomorphic to real numbers, and the pairs of number (0, b) are not isomorphic to real numbers. All the three ways may not be very meaningful to students. However, the effectiveness of the ways in the development of complex numbers remains unchanged. The students learn the complex numbers through ways established by real situations. Nowadays, the following representations are used for complex numbers:

- ***i.*** Points or vectors in the plane
- ***ii.*** Ordered real number pairs
- ***iii.*** Movements such as vector displacement or rotation in the plane
- ***iv.*** Numbers in the form of a + ib
- ***v.*** Real-coefficient polynomials,
- ***vi.*** $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ matrices in the form
- ***vii.*** Algebraically a complete and closed field

1.2.Visual Representations

Arcavi(2003) describes visualization as the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. By considering visualise a problem means to understand the problem in terms of a visual image, Presmeg(1992) describes visualization process as one that involves visual representations, with or without a diagram, as an essential part of the method of solution. Representation and visualization are at the core of understanding in mathematics. Visual representation, especially diagrams, are ubiquitous and can facilitate problem solving in mathematics. Representing mathematical thinking is an important problem-solving process; students should be able to ‘‘use representations to model and interpret physical, social, and mathematical phenomena’’(NCTM 2000, p.206). Representations can take a variety of forms, including pictures written symbols, manipulative models, oral language, and real-world situations. Schnotz(2002) suggests that symbols or texts and visual displays belong to different classes of representations, namely descriptive and depictive representations, respectively.

Depictive or visual representations include iconic signs that are associated with the content that they represent through common structural features at either a concrete or more abstract level. In the field of mathematics learning and instruction, visual representations play an important role as a means to communicate mathematical ideas. Visual representations are defined as a collection of graphical symbols encoding properties and relationships for a represented world consisting of mathematical structures or concepts (Cuoco and Curcio, 2001). In this paper, the term “representations,” in a restricted form, is interpreted as the tools used for representing mathematical ideas, such as diagrams and equations. The abstract arguments presented in Euclid's Elements heavily rely on the use of diagrams, and this use of visual representations remained an acceptable practice in mathematics well into the eighteenth century (Stylianou&Silver, 2004).

1.3.Complex Numbers and Visual Representations

In 1673, John Wallis introduced the concept of complex number as a geometric entity, and more specifically, the visual representation of complex numbers as points in a plane (Steward and Tall, 1983, p.2). The representation comprised a horizontal line as a real part of the complex numbers and another line perpendicular to the line as an imaginary part (Figure 1).

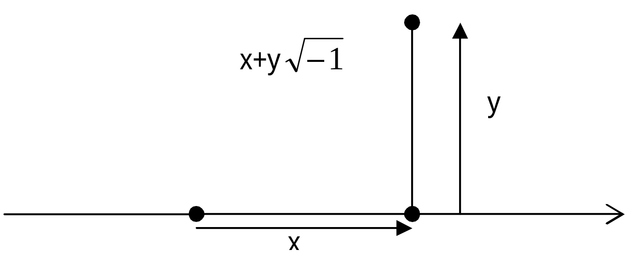

Figure 1. Wallis’s visual representation related complex numbers

Subsequently, Caspar Wessel and Jean Robert Argand developed appropriate visual representations of complex numbers. Nevertheless, the visual representations, even at that time, were not accepted with enthusiasm. Complex numbers are very closely associated with the geometrical interpretation of ordinary complex numbers as points of a plane, which apparently was first mentioned by the Danish, K. Wessel, but became widely known chiefly through the studies of the famous mathematicians, K.F. Gauss and A. Cauchy. Furthermore, Euler visualized complex numbers using a geometrical representation in polar coordinates.

Visual representations of complex numbers is basically based on trigonometry. The number $z = x+iy$ corresponds to the point (x, y) in the plane. By contrast, any point/each real numbers pairs in the plane corresponds to a complex number. Owing to this characteristic of complex numbers, each complex number can be expressed in the form of $z=x+iy=(x,y)$ (Figure 2).

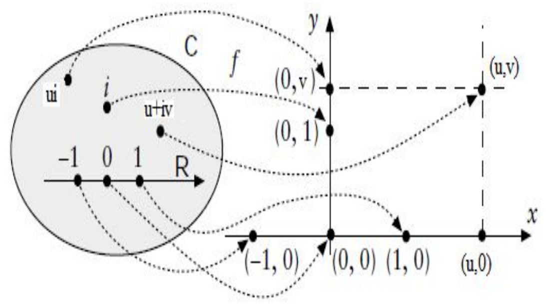

Figure 2. The relation between sets C and RxR

As each complex number is represented with an ordered pair, a vector is obtained in combined (0,0) origin point and (x, y) ordered pair representing z = x + iy. Therefore, the complex number z is called a vector. Considering a complex number as a vector is the basic scaffolding for the module and argument concepts. By helping argument of a vector corresponding to the complex number, one can make geometric interpretations deal with multiplication and division of the numbers.

Geometrically, multiplication by a complex number A=[r,ϕ] is the rotation of the plane through angle ϕ and an expansion of the plane by factor r. Several features of the movements are as follows:

i. Both the rotation and expansion are centred at the origin.

ii. It makes no difference whether we do the rotation followed by the expansion or the expansion followed by the rotation.

iii. If r < 1 then the expansion is in a contraction (Neddham, 1997).

In Figure 3, it is possible to observe the effects of such a transformation.

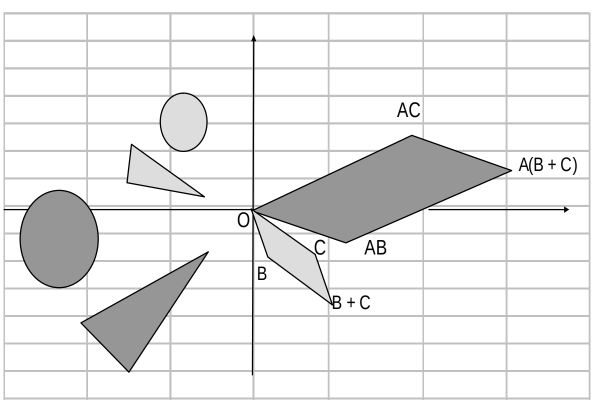

Figure 3. Rotation and expansion in the plane

Sfard(1991) assumes that good visual representations may help learners to progress from an 'operational' to 'structural' understanding. Strong geometric connection and interpretations are required to consider the complex numbers as points in the plane. The equation

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$$

can be used as an example to show the interaction with the geometry of complex numbers. Although a long process is algebraically required for proving the equality

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$$

a simple and elegant proof can be obtained through visual representation. In the application, which is the most important example for a strong link between Euclidean geometry and complex numbers, a well-known property that deals with the parallelogram has been proved completely by geometric interpretations through four nonlinear points in the plane.

Diagonals of parallelogram OPRQ are geometrically

$$|OR| = |z + w|, |QP| = |z - w|$$

Consequently, the equality

$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$ can be visually determined by the theorem, the sum of the squares of the diagonals in any parallelogram is twice the sum of the squares of the sides(Figure 4).

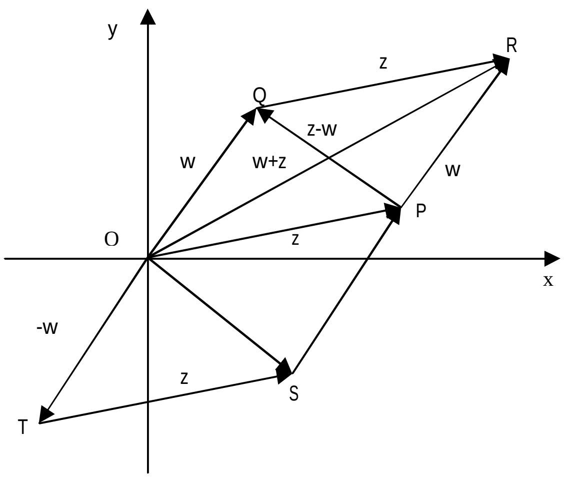

Figure 4. The visual representation of

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$$

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