Bayesian Hypothesis Testing−Comparison of Two Procedures

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Abstract

The paper deals with the comparison of the two known procedures of the Bayesian one-sided hypothesis testing. First procedure is based on the simple comparison of the probabilities of null and alternative hypotheses, second is based on the comparison of the probability of null hypothesis and the chosen significance level. The procedures are compared on the basis of the probability that the rejected hypothesis is true. The using of the first named procedure is recommended on the basis of realized analysis, in the case of one-sided testing.

Key words: Bayesian hypothesis testing, one-sided testing, two-sided testing, posterior distribution

1. Introduction

Two procedures of the Bayesian hypothesis testing will be compared in this paper. The comparison will be illustrated on the example of a testing hypothesis about a normal mean.

In classical hypothesis testing, a null hypothesis $H_0 : \theta \in \Theta_0$ and a alternative hypothesis $H_1 : \theta \in \Theta_1$ (θ is an unknown parameter) are specified. A test procedure is evaluated in terms of the probabilities of type I and type II error (Miller, Miller, 2004; Parmigiani, Inoue, 2009). In Bayesian analysis one merely calculates the posterior probabilities $\alpha_0 = P(\Theta_0|x)$ and $\alpha_1 = P(\Theta_1|x)$ and decides between H_0 and H_1 accordingly (Berger, 1993, p. 145, Lee, 1989, p. 124-125). In both approaches, $\Theta_0 \cup \Theta_1 = \Theta$ and $\Theta_0 \cap \Theta_1 = \varnothing$.

2. One-Sided Testing

One-sided hypothesis testing occurs when $\Theta \subset R^1$ and Θ_1 is entirely to one side of Θ_0. Let\` s have the one-sided hypothesis tests about a normal mean:

$$H_0 : \mu \le \mu_0 \quad (1)$$
$$H_1 : \mu > \mu_0 \quad (2)$$

or

$$H_0 : \mu \ge \mu_0 \quad (3)$$
$$H_1 : \mu < \mu_0 \quad (4)$$

where μ_0 is a known constant.

According to the described procedure, we can simply calculate the posterior probabilities $P(H_0)$, $P(H_1)$ and if $P(H_1) \ge P(H_0)$, reject H_0 and accept H_1. We will entitle this procedure – **Procedure A**.

The second proposition is following. The calculation the posterior probability of the null hypothesis $P(H_0)$ is realized. If this probability is less than chosen significance level α, the null hypothesis is rejected and H_1 can be accepted (Bolstad, 2004, p. 201). We will entitle this procedure – **Procedure B**.

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3. Comparison of the Procedures A and B

It is clear that the procedure A is equivalent to the procedure B for the significance level $\alpha = 0.5$.

Now, the cases when $\alpha < 0.5$ will be studied. We will study (1) – (2) and evaluate the effectiveness of the procedures according to the following probability of error:

$P(H_r) = P$(rejected hypothesis is true) for different values of μ_0.

Suppose, we have the normal posterior distribution of the mean μ with the median $\mu^*_{\frac{1}{2}}$ and known standard deviation σ^*.

Let` s take **Procedure A**:

- For $\mu_0 \le \mu^*_{\frac{1}{2}}$ the hypothesis H_0 is rejected. Then $P(H_r) = P(H_0)$.
- For $\mu_0 > \mu^*_{\frac{1}{2}}$ the hypothesis H_1 is rejected and $P(H_r) = P(H_1)$.

Now, we will study **Procedure B**:

- For $\mu_0 \le \mu_\alpha$, where μ_α is the $(1 - \alpha)$-quantile of the posterior distribution, the null hypothesis will be rejected and we have $P(H_r) = P(H_0)$.
- For $\mu_0 > \mu_\alpha$ the null hypothesis will be accepted and we have $P(H_r) = P(H_1)$.

We can see that:

- For $\mu_0 \le \mu_\alpha$, the value of $P(H_r) = P(H_0)$ is the same for both procedures.
- For $\mu_0 \in [\mu_\alpha, \mu^*_{\frac{1}{2}}]$, $P(H_r) = P(H_0)$ for procedure A and $P(H_r) = P(H_1)$ for procedure B (for these values of μ_0, $P(H_0) \le P(H_1)$).
- For $\mu_0 > \mu^*_{\frac{1}{2}}$ the value of $P(H_r) = P(H_1)$ is the same for both procedures

It is clear that generally, for fixed $\alpha < 0.5$, the probability of error $P(H_r)$ of procedure A is less or equal than that of procedure B for $\mu_0 \in [\mu_\alpha, \mu^*_{\frac{1}{2}}]$ and the same for others values of μ_0. So, generally, procedure A should be preferred.

It is also clear that this conclusion can be generalized for the test (3) – (4), the tests about another parameters and for another continuous posterior distributions.

Two-sided hypothesis testing occurs when $\Theta \subset R^1$ and Θ_1 is to both sides of Θ_0:

$H_0 : \theta_0 \le \theta \le \theta_1 \qquad (\theta_0 < \theta_1)$

$H_1 : \theta > \theta_1$ or $\theta < \theta_0$ where θ_0, θ_1 are known constants.

Procedure A can be used also for the two-sided hypothesis testing.

4. Conclusion

We suggest the using of the procedure A in the case of one-sided or two-sided testing. The analysis made on the basis of minimization of the probability that rejected hypothesis is true showed that according to this criteria, procedure A is ever better then procedure B. In our opinion, the procedure A is generally more consistent with the rational decision making.

When we want to conduct a test of a point null hypothesis, the Lidley` s method can be of interest. In Lee, 1989, p. 131 is recommended to use it in situations where there are several unknown parameters and the complete posterior is difficult to describe or take in.

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