Scattering by a Perfectly Electromagnetic Conducting Random Grating

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Abstract

An analytic theory for the electromagnetic scattering from a perfectly electromagnetic conductor (PEMC) random grating is developed, using the duality transformation which was introduced by Lindell and Sihvola. The theory allows for the occurrence of cross-polarized fields in the scattered wave, a feature which does not exist in standard scattering theory. This is why the medium is named as PEMC. PEMC medium can be transformed to Perfect electric conductor (PEC) or perfect magnetic conductor (PMC) media. As an application, plane wave reflection from a planar interface of air and PEMC medium is studied. PEC and PMC are the limiting cases, where there is no cross-polarized component.

1 Introduction

The problems we are considering, i.e., scattering from half plane, strip or grating are very well known in the field of electromagnetics. Our aim is not to resolve these problems but introduce few random parameters in these planner boundaries [1, 2, 3, ?]. A complete solution exists for the perfectly electric conducting case in literature, based on the following equations and conditions, and to study the effects of the stochastic nature of these boundaries on the scattered field. Before examine the random boundaries, i.e., scatterers with random parameters it is instructive to examine the behavior of random grating, because in two dimensional planner perfectly conducting boundaries, with sharp edges. An effort has been made to transform average field from pec case to pemc case.

2 Formulation

Consider a random grating, random distribution of screens and gaps along xz-plane [1], where L_n is the length of nth screen and l_n is that gap of nth gap. The length L_n of screen and l_n of gap are random variables with certain distributions.

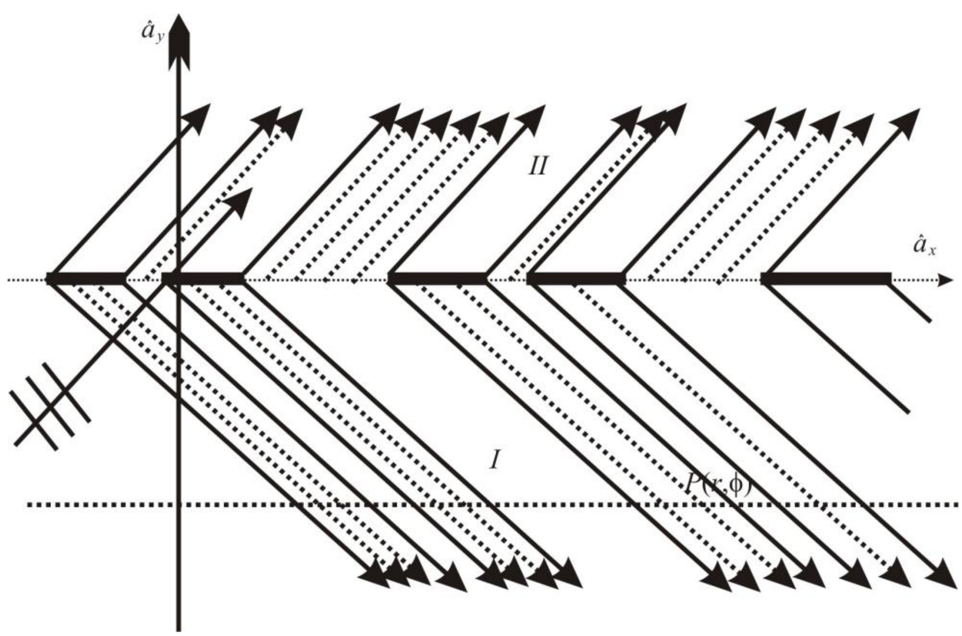

Figure 1: Geometry of the problem

Let us assume that a uniform plane wave is incident upon this random grating, shown in the Fig.(1). The incident field could be written as

$$E^i = \hat{a}_z e^{ikr\cos(\phi^r-\phi)} = \hat{a}_z e^{ik(x\cos\phi^i - y\sin\phi^i)} \quad (1)$$

where we assumed unit amplitude wave and $0 < \phi^i < \pi$

Simply need to know the probabilistic nature of this random boundary,the scattered field will also be random. The main objective is to calculate statistics of scattered field, at least up to second order.

Suppose that length l_n of screen and l_n of gap are distributed exponentially and statistically independent i.e., i.i.e. random variables. Exponential distribution is chosen due to memoryless nature of random grating. Hence, they can be modelled as Poisson,s process. The average length of screen and gap are presented by $< L >$ and $< l >$ respectively. The probability density functions of these random screens and gaps are given by,

$$P_L(L) = \frac{1}{<L>} e^{-\frac{L}{<L>}} \quad (2)$$

$$P_l(l) = \frac{1}{<l>} e^{-\frac{l}{<l>}} \quad (3)$$

fig.(..).

The scattered GO field at the receiving point $P(r,\phi)$ can be written as

$$E^s(P) = E^{I,II} + E^d \quad (4)$$

where $E^{I,II}$ represents the field reflected in the both regions and E^d is the diffracted field due to edge contribution. Similarly, we can write the total field in the region II, i.e.,$y > 0$ at the observation point $P_1(r,\phi)$ as,

3 Statistics of the Scattered Field

The scattered field in the region I and in the region II is given by equations

$$E^s(P) = f(x,y)E^r + \sum_j E^{d(j)} y < 0 \quad (5)$$

$$E^s(P_1) = g(x,y)E^i + \sum_j E^{d(j)} y < 0 \quad (6)$$

where E^r is field reflected by the wall and is given by

$$E^r = \mathbb{R}e^{\imath krcos\,(\phi^r-\phi)} \quad (7)$$

and $f(x,y)$ is either 1 or 0, depending upon observation point $P(r,\phi)$, lies in the region of specular reflection point or no reflected. $E^{d(j)}$ represents the contribution of jth edge in the total diffracted field. E^i is the field incident on the wall and $g(x,y)$ has similar behavior as $f(x,y)$ but $g(x,y) = 0$ where $f(x,y) = 1$ and vice versa. That is $g(x,y)$ is the complementary of $f(x,y)$ function. Now the average scattered field in the two regions can be written as

$$< E^s >=< E^{ref} > + < E^d > y < 0 \quad (8)$$

$$< E^s >=< E^t > + < E^d > y > 0 \quad (9)$$

and the variance of these fields could be written as

$$var(E^s) = var(E^{ref}) + var(E^d) + 2\Re cov(E^{ref}, E^d) y < 0 \quad (10)$$

$$var(E^s) = var(E^t) + var(E^d) + 2\Re cov(E^t, E^d) y > 0 \quad (11)$$

4 Average Reflected Field and its Variance

The reflected field at point $P(r,\phi)$ in the region I, could be written as

$$E^{ref}(r,\phi) = f(x,y)E^r(r,\phi) \quad (12)$$

where E^r, as given

$$E^r(r,\phi) = \mathbb{R}e^{\imath krcos\,(\phi^r-\phi)} \quad (13)$$

and $f(x,y)$ could be modelled by telegraph signal process $f(x)$ defined by

$$f(x,y) = f(x) = \{1, ifxliesonthescreenifyifxliesontheslit \quad (14)$$

This process is statistically stationary, i.e., independent of shift origin. The function $f(x,y)$ is independent of y due to stationary nature of this function. Now average reflected field as,

$$< E^{ref} >=< f(x) > E^r \quad (15)$$

where $< f(x) >$ is the average value of the telegraph process.

$$< f(x) >= \frac{<L>}{<L>+<l>} = M \quad (16)$$

$$var(f(x)) = \frac{<L><l>}{<L>+<l>}^2 = M(1-M) \quad (17)$$

where $< L >$ and $< l >$ are the average screen and slit lengths respectively. Hence final expression for the average reflected field will become,

$$< E^{ref} >= \frac{-<L>}{<L>+<l>} e^{\imath krcos\,(\phi^r-\phi)} \quad (18)$$

and the variance of the reflected field could be written as

$$var(E^{ref}) =< f^2(x) > -< f(x) >^2 = M(1-M) = \frac{<L><l>}{<L>+<l>}^2 \quad (19)$$

From the above equations it is clear that on the average the reflected field to be plane wave whose variance is a constant and depends on the average parameters of random grating, i.e., average length of screens and gaps. For large $< L >$, variance decays and approaches to zero.

5 Average Transmitted Field and its Variance

In the region II, for $y > 0$, the scattered field can be written as the incident field which has passed through the random grating and the diffracted field due to edges, where E^t at point $P_1(r,\phi)$ is given by

$$E^t(r,\phi) = g(x,y)E^i(r,\phi) \quad (20)$$

where $g(x,y)$ could be modelled by component of telegraph process, i.e.,

$$g(x,y) = g(x) = f^c(x) \quad (21)$$

that is the effect of screen and slit is interchanged.
The average value of the transmitted field in the region II given by,

$$< E^{II}(r,\phi) >=< g(x,y) > E^i(r,\phi) = \frac{<L>}{<L>+<l>} \quad (22)$$

The average transmitted field and its variance as,

$$< E^t >=< g(x) > E^i = (\frac{<l>}{<L>+<l>})e^{ikrcos\,(\phi^i-\phi)} \quad (23)$$

$$var(E^t) =< |E^t|^2 > -| < E^t > |^2 = \frac{<l><L>}{(<L>+<l>)^2} \quad (24)$$

Again the average transmitted field is a plane wave with constant variations, and for much larger average screen lengths compared to gaps the average transmitted field goes to zero and its variance also approaches to zero.

6 Average Diffracted Field and its Variance

Now we will calculate the average diffracted field and its variance. The random grating has gaps distributed along its length, therefore at every transition point between screen and gap or vice versa, we have an edge and the field diffracted from it will contribute towards the total scattered field. To determine the location edges we need their statistical distributions, which will be derived from telegraph signal process $f(x)$. The edges on the screen are represented by transitions between 0 and 1 and 1 and 0 in $f(x)$. To represent these edges explicitly the telegraph signal is differentiated as

$$|f'(x)| = \sum_i \delta(x - x_i) \quad (25)$$

As the lengths of individual screen and gap follow an exponential distribution, the transitions represented by Dirac delta functions. The positive sign for trailing wall edges and -ive for leading wall edges. The diffracted field from two types of edges will be calculated. To calculate the average rate of occurrence of the edge, i.e., $f'(x) > 0$. Here, one edge is for each set of screen and gap and the average length of a screen and gap is given by $< L >$ and $< l >$ respectively. Similarly, there is one trailing edge for each set of screen and gap. The Poisson parameter for the edge is given by [2]

$$\lambda = \frac{2}{<L>+<l>} \tag{26}$$

The total diffracted field will be the sum of contributions from all such edges, half of these edges will be the leading edges and other half will be trailing edges.

Consider Far-zone approximation and assuming, they are independent from each other.

The diffracted field due to all edge contributions could be written as,

$$E^d = 2C \sum_j H_o^1(kR) e^{ik x_j \cos\phi^i} \tag{27}$$

where $R = \sqrt{(x - x_j)^2 + y^2}$ and $C = \frac{-1}{4}$

In the above equations x'_j are random points making the Poisson point process. Take the statistical expectation on both sides of above equation, the expression becomes as

$$< E^d >= 2C \sum_j (< H_o^1(kR) e^{ik x_j \cos\phi^i} >) \tag{28}$$

Also it can written as convolution of two functions as,

$$E^d = 2Cs(x) = 2Ch(x) \otimes g(x)^3 \tag{29}$$

where $h(x) = H_o^1(kr)$ where $r = \sqrt{x^2 + y^2}$
and $s(x) = q(x) \sum_j \delta(x - x_j)$
The average diffracted field can be calculated,

$$< s(x) >= \lambda \int_{-\infty}^{\infty} H_o^1(kR) e^{\imath k\alpha \cos\phi^i} d\alpha \tag{30}$$

where $R = \sqrt{(x - \alpha)^2 + y^2}$

$$< E^d >= 2C < s(x) > \tag{31}$$

The average diffracted field can be written as,

$$< E^d >= (\frac{-\lambda}{2}) \int_{-\infty}^{\infty} H_o^1(kR) e^{\imath k\alpha \cos\phi^i} d\alpha \tag{32}$$

Consider a uniform plane wave is incident on a perfectly conducting xz-plane at an angle ϕ^i,
The incident field E^i will induce a surface current on the plane given by [3]

$$J_s(x)\hat{a}_z = \frac{2k}{(\omega)(\mu)} Sin(\phi^i) e^{\imath kx\cos(\phi^i)} (\hat{a}_z) \tag{33}$$

This induced surface current density give rise to reflected field as given by the relation

$$E_z^r(r,\phi) = -e^{\imath kr\cos(\phi^r - \phi)} = \int_{-\infty}^{\infty} J_s(x') G(r; x') d(x') \tag{34}$$

where $G(r; x')$ is the *Green,s function* given by,

$$G(r; x') = \frac{-\omega\mu}{4} H_o^1(kR) \tag{35}$$

where $R = \sqrt{(x - x')^2 + y^2}$
Now using current expression, we get

$$e^{\imath kr\cos(\phi^r - \phi)} = \frac{k}{2} sin(\phi^i) \int_{-\infty}^{\infty} e^{\imath kx'} cos(\phi^i) H_o^1(kR) dx' \tag{36}$$

Applying the identity

$$\int_{-\infty}^{\infty} e^{ikx'\cos(\phi^i)} H_o^{(1)}(kR) dx' = \frac{2\lambda}{k\sin(\phi^i)} e^{ikr\cos(\phi^r - \phi)} \int_{-\infty}^{\infty} x'\cos(\phi^i) H_o^1(kR) dx' = \frac{2\lambda}{k\sin(\phi^i)} e^{ikr\cos(\phi^r - \phi)} \tag{37}$$

The average diffracted field can be written as,

$$< E^d >= \frac{-\lambda}{k\sin(\phi^i)} e^{\imath kr\cos(\phi^r - \phi)} \tag{38}$$

therefore expression for variance of $s(x)$

$$var(s(x)) = \lambda \int_{-\infty}^{\infty} H_o^1(kR) H_o^2(kR) d(\alpha) \tag{39}$$

where $R = \sqrt{(x-\alpha)^2 + y^2}$

and expression for diffracted field can be written as,

$$var(E^d) = \frac{\lambda}{4} \int_{-\infty}^{\infty} H_o^1(kR) H_o^2(kR) d(\alpha) \tag{40}$$

The integral in the above equation is not convergent, therefore second moment of variant or the variance of diffracted field is not finite. This is due to the fact that in the averaging we have considered every possible realization of the Poisson points and on the average there effect tends to a plane wave but there is infinite power in the plane wave. We can model the statistics of diffracted field by Heavy Tail distributions, because its mean is finite but second moment is infinite. The above field can be transformed from perfectly electric conducting case to perfectly electromagnetic conducting case by the concept of PEMC introduced by Lindell and Sihvola [3, ?] is a generalization of both PEC and PMC. An analytic theory for the electromagnetic scattering from a PEMC plane where a line source has been placed randomly, is developed.

The PEMC medium characterized by a single scalar parameter M, which is the admittance of the surface interface, where $M = 0$ reduces the PMC case and the limit $M \to \pm\infty$ corresponds to the perfect electric conductor (PEC) case. The theory allows for the occurrence of cross-polarized fields in the scattered wave in the scattered wave, a feature which does not exist in standard scattering theory. This means that PEC and PMC are the limiting cases, for which there is no cross-polarized component. Because the PEMC medium does not allow electromagnetic energy to enter, an interface of such a medium behaves as an ideal boundary to the electromagnetic field. At the surface of a PEMC media, the boundary conditions between PEMC medium and air with unit normal vector n, are of the more general form. Because tangential components of the E and H fields are continuous at any interface of two media, one of the boundary conditions for the medium in the air side is $n \times (H + ME) = 0$, because a similar term vanishes in the PEMC-medium side. The other condition is based on the continuity of the normal component of the D and B fields which gives another boundary condition as $n.(D - MB) = 0$.
Here, PEC boundary may be defined by the conditions

$$n \times E = 0, \qquad n.B = 0 \tag{41}$$

While PMC boundary may be defined by the boundary conditions

$$n \times H = 0, \qquad n.D = 0 \tag{42}$$

where M denotes the admittance of the boundary which is characterizes the PEMC. For $M = 0$, the PMC case is retrieved, while the limit $M \to \pm\infty$ corresponds to the PEC case. Possibilities for the realization of a PEMC boundary have also been studied [4]. It has been observed theoretically that a PEMC material acts as a perfect reflector of electromagnetic waves, but differs from the PEC andthe PMC in that the reflected wave has a cross-polarized component. The duality transformations of perfectly electric condutor (PEC) to PEMC have been studied by many researchers [3, ?, 4, 5, 6, 7, 8]. Here we present an analytic scattering theory for a PEMC step, which is a generalization of the classical scattering theory.

Applying a duality transformation which is known to transform a set of fields and sources to another set and the medium to another one. In its most general form, the duality transformation can be defined as a linear relation between the electromagnetic fields. The effect of the duality transformation can be written by the following special choice of transformation parameters:

$$\begin{pmatrix} E_d \\ H_d \end{pmatrix} = \begin{pmatrix} M\eta_0 & \eta_0 \\ \frac{-1}{\eta_0} & M\eta_0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \tag{43}$$

has the property of transforming PEMC to PEC, while

$$\begin{pmatrix} E \\ H \end{pmatrix} = \frac{1}{(M\eta_0)^2 + 1} \begin{pmatrix} M\eta_0 & -\eta_0 \\ \frac{1}{\eta_0} & M\eta_0 \end{pmatrix} \begin{pmatrix} E_d \\ H_d \end{pmatrix} \tag{44}$$

has the property of transforming PEC to PEMC [?].

Following the above relations [3], the transformed equations becomes as

$$E^r = -\frac{1}{M^2\eta_0^2+1}\left[(-1+M^2\eta_0^2)E^i + 2M\eta_0 u_z \times E^i\right] \quad (45)$$

$$E_{sd} = -(M\eta_0 E_s + \eta_0 H_s) \quad (46)$$

$$H_{sd} = -\frac{1}{\eta_0}E_s + M\eta_0 H_s \quad (47)$$

$$E_s = \frac{1}{(M\eta_0)^2+1}\left[M\eta_0 E_{sd} - \eta_0 H_{sd}\right] \quad (48)$$

$$E_s = \frac{1}{(M\eta_0)^2+1}\left[((M\eta_0)^2-1)E_s - 2M\eta_0^2 H_s\right] \quad (49)$$

$$E_s = \frac{1}{(M\eta_0)^2+1}\left[((M\eta_0)^2-1)E_s - 2M\eta_0 E_s\right] \quad (50)$$

Where E_s, H_s are transformed pemc average fields and E_{sd}, H_{sd} are average scattered elecric and magnetic fields respectively. This means that, for a linearly polarized incident field E^i, the reflected field from a such a boundary has a both co-polarized component, while $u_z \times E^i$ is a cross-polarized component, in the general case. For the PMC and PEC special cases ($M = 0$ and $M = \pm\infty$ respectively), the cross-polarized component vanishes. For the special PEMC case $M = \frac{1}{\eta_0}$, such that

$$(E^r = -u_z \times E^i) \quad (51)$$

which means that the reflected field appears totally cross-polarized. It is obvious theoretically that a PEMC material acts as a perfect reflector of electromagnetic waves, but differs from the PEC ($E^r + E^i) = 0$ and $H^r = H^i$) and PMC ($E^r = E^i$ and $H^r + H^i = 0$) in that the reflected wave has a cross-polarized component.

7 Concluding remarks

In this work, a plane wave scattering by a Perfectly Electromagnetic Conducting random grating has been studied. The theory provides explicit analytical formulas for the electric and magnetic field. An other formulla has been derived for the relative contributions to the scattered fields of the co-polarized and the crosspolarized fields depend on parameter M. The cross-polarized scattered fields vanish in the PEC and PMC cases, and are maximal for $M = \pm 1$. In the general case,the reflected wave has both a co-polarized and a cross-polarized component. The above transformed solution presents an analytical theory for the scattering by randomly placed perfectly electromagnetic conducting random grating. It is clear from the above discussion that for $M \to \infty$ and $M \to 0$ correspond to the PEC and PMC respectively. Moreover, for $M = \pm 1$ the medium reduces to PEMC.

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